Chapter 7. Polarization Optics - Jones Matrix

The optics of LCD is complicated by the fact that it is birefringent as well as electroactive (with a twist).

The simplest approach of modeling LCD optics is to use the 2x2 matrix. (Jones matrix). The LC cell is characterized by $\theta(z)$ and $\phi(z)$. Once these two functions are known, the optical properties of the LCD can be calculated.

7.1 2x2 matrix

The 2x2 Jones matrix is just a simple shorthand way to represent the polarization state of light. Since LCD is based on polarization manipulation, the Jones matrix is very useful.

The polarization state of light is described by a 2x1 vector (Jones vector). Any polarization state can be represented as a sum of two perpendicularly polarized waves with different amplitude and phase:

$$E = (xa + yb\, e^{j\delta})E_0\, e^{j\omega t - jkz}$$

where $|a|^2 + |b|^2 = 1$, and $\delta$ is the phase delay between the $x$ and $y$ components. The Jones vector corresponding to this wave is

$$\begin{pmatrix} a \\ b e^{j\delta} \end{pmatrix}$$
Notice that any common phase in $a$ and $b$ can be taken out and be absorbed by the phase term $e^{j\omega t-jkz}$.

Examples of Jones vectors:

\[
\begin{pmatrix}
\cos \alpha \\
\sin \alpha
\end{pmatrix}
\]
refers to light polarized at angle $\alpha$ to the $x$-axis.

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ j \end{pmatrix}
\]
represents a right-circularly polarized light.

In the Jones vector and Jones matrix approach, every optical element is represented by a 2x2 matrix.

\[
\begin{pmatrix} J_{\text{in}} \\ \hline \\
M \\
\hline \\
J_{\text{out}} \end{pmatrix}
\]

Then \( J_{\text{out}} = M J_{\text{in}} \)

A few examples of Jones matrices:

\[
\text{x-axis polarizer} \hspace{1cm} M_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]
Retardation plate with y-axis as the fast axis:

\[ M_{\delta} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{j\delta} & 0 \\ 0 & e^{-j\delta} \end{pmatrix} \]

Polarization rotator:

\[ M_{R}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]

Can easily show that a linearly polarized light polarized at \( \alpha \) to the x-axis becomes \( \alpha + \theta \) to the x-axis after passing through \( M_3 \).

\[ \begin{pmatrix} \cos(\alpha + \theta) \\ \sin(\alpha + \theta) \end{pmatrix} = M_{R}(\theta) \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \]

The Jones matrix is very useful in problems concerning polarization manipulation, such as the LCD. If there are more than one polarization manipulation element, we simply multiply the Jones matrices:

\[ J_{\text{in}} \rightarrow M_1 M_2 M_3 M_N J_{\text{out}} \]
and $\phi$ is the rotation angle from $(x,y)$ to $(x',y')$ as shown in the diagram.

Note that $R_\phi = M_{R^{-1}}(\phi) = M_{R}(-\phi)$

Also $R_{-\phi} = R_{\phi}^{-1} = R_{\phi}'$

Proof:

It is easy to show that

$$\begin{pmatrix} x \\ y \end{pmatrix} = R_{\phi}^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} = M_R(\phi) \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Also, any vector $J$ in $(x, y)$ system is related to $J'$ in $(x', y')$ system by a rotation

$$J = R_{-\phi} J'$$

or

$$J' = R_{\phi} J$$

Hence, if $K = M J$

then $K' = R_{\phi} K = R_{\phi} M J = R_{\phi} M R_{-\phi} R_{\phi} J$

or

$$K' = M' J'$$

where $M' = R_{\phi} M R_{-\phi}$

Also the reverse transformation is given by

$$M = R_{\phi}^{-1} M' R_{\phi}$$
For example, rotate x axis by \(+90^\circ\),

\[
R_{90} = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]

A retardation plate with x-axis as fast axis will have a Jones matrix of

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
ey^j\delta & 0 \\
0 & ey^{-j}\delta
\end{pmatrix}
\begin{pmatrix}0 & -1 \\
1 & 0
\end{pmatrix}
= \begin{pmatrix}ey^{-j}\delta & 0 \\
0 & ey^j\delta
\end{pmatrix}
\]

The Jones matrix of a retardation plate with the fast axis at \(\phi\) to the x-axis is given by

\[
R_{-\phi} \begin{pmatrix}e^{-j}\delta & 0 \\
0 & e^{j}\delta\end{pmatrix} R_{\phi} = \begin{pmatrix}\cos\phi & \sin\phi \\
\sin\phi & \cos\phi\end{pmatrix} \begin{pmatrix}e^{-j}\delta & 0 \\
0 & e^{j}\delta\end{pmatrix} \begin{pmatrix}\cos\phi & -\sin\phi \\
\sin\phi & \cos\phi\end{pmatrix}
\]

Note that we are transforming from \((x',y')\) to \((x,y)\) in this case. The \((x',y')\) frame is the \textbf{principle axes} of the retardation plate and the \((x,y)\) axes are the fixed \textbf{laboratory frame}.

This formula is very important. We shall see later that the LC cell can be regarded as a stack of birefringent plates.

Some more manipulations of Jones matrices:

1. Half wave plate:

The Jone matrix of a half wave plate with c-axis on the x-axis is

\[
\begin{pmatrix}
-j & 0 \\
0 & j
\end{pmatrix}
\]
Then \[ J_{\text{out}} = M_N \ldots M_3 M_2 M_1 J_{\text{in}} \]

### 7.2 Coordinate transformation

The Jones matrix depends on the definition of the coordinate system. If the coordinate is rotated by \( \phi \), the Jones matrix will become different.

![Coordinate transformation diagram](image)

**Rule:** If the Jones matrix is \( M \) in the (x,y) coordinates and \( M' \) in the (x',y') coordinate system, then

\[
M' = R_\phi M R_\phi^{-1}
\]

and

\[
M = R_\phi^{-1} M' R_\phi
\]

where \( R_\phi \) is the coordinate transformation matrix

\[
R_\phi = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}
\]
The Jones matrix of a half wave plate with c-axis at $\theta$ to the x-axis is

$$\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} -j \ 0 \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} = \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}$$

Check:

$$\begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos 2\phi \\ \sin 2\phi \end{pmatrix}$$

which is equivalent to a rotation of the linear polarization along x by $2\phi$. This Jones matrix is not the same as the polarization rotation matrix since the rotation is dependent on the polarizer angle.

2. Quarterwave plate

The Jones matrix of a quarterwave plate with c-axis along the x-axis

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1-j & 0 \\ 0 & 1+j \end{pmatrix}$$

If light polarized at $45^\circ$ to the x-axis passes through it, the new Jones vector is

$$J = \frac{1}{2} \begin{pmatrix} 1-j & 0 \\ 0 & 1+j \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1-j \\ 1+j \end{pmatrix} = \frac{1-j}{2} \begin{pmatrix} 1 \\ j \end{pmatrix}$$

which is a right circularly polarized wave.

3. Polarizer
The Jones matrix of a polarizer with polarizing axis along $x$ is

$$\begin{pmatrix}
1 & 0 \\
0 & 0 \\
\end{pmatrix}.$$  So the Jones matrix of a polarizer with the polarizing axis at $\theta$ is given by

$$\begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi \\
\end{pmatrix}
= \begin{pmatrix}
\cos^2 \phi & \sin \phi \cos \phi \\
\sin \phi \cos \phi & \sin^2 \phi \\
\end{pmatrix}$$

4. Eigenvalues and eigenvectors of the Jones matrix

Any matrix can be diagonalized to find the eigenvalues and eigenvectors. For the 2x2 Jones matrix, the 2 eigenvectors correspond to the Jones vector that can propagate through the system without any change of polarization state.

$$M J = \lambda J$$

Exercise: Find the eigenvectors of the retardation plate and the polarizer Jones matrices.

7.3 LCD Optics Modeling

In the most common model, the LC cell is thought of as composed of N retardation plates. N is sufficiently large so that each slice can be regarded as having constant $\theta$ and $\phi$, i.e. constant c-axis orientation. For twist angle smaller than 180°, N<80 is large enough for LCD modeling. Also N
> 20 for accuracy. This is the approach used in all commercial LCD modeling software.

The LC cell then has a Jones matrix given by

$$M_{LC} = M_N \ldots M_3 M_2 M_1$$

where $M_n$ is the Jones matrix of a birefringent plate with c-axis at angle $\phi_n$ to the x-axis and at $\theta_n$ to the z-axis.

![Diagram](image)

**Jones matrix $M_n$**

If the fast axis is at angle $\phi_n$ to the x-axis, then the Jones matrix is given by a coordinate transformation:

$$M_n = R_{\phi_n}^{-1} M' R_{\phi_n}$$

where $R_{\phi_n}$ is the transformation matrix
\[
R_{\phi_n} = \begin{bmatrix}
\cos \phi_n & \sin \phi_n \\
-\sin \phi_n & \cos \phi_n
\end{bmatrix}
\]

Jones matrix of the general birefringent plate with c-axis along the principle axis \((\mathbf{x}')\) is given by

\[
M' = \begin{pmatrix}
e^{-j\delta} & 0 \\
0 & e^{j\delta}
\end{pmatrix}
\]

where \(\delta = \frac{\pi d (n_e(\theta) - n_o)}{\lambda}\)

with

\[
\frac{1}{n_e^2(\theta)} = \frac{\cos^2 \theta}{n_o^2} + \frac{\sin^2 \theta}{n_e^2}
\]

The above formulas can be used for the modeling of LCD with arbitrary \(\phi(z)\) and \(\theta(z)\) distribution numerically. In particular, we shall work out a simple case of \(\phi(z)\) and \(\theta(z)\) below.

7.4 Jones matrix of twisted nematic cells with uniform tilt

The Jones matrix of a T-cell without any voltage applied can be obtained analytically.

Assuming no pretilt for simplicity, \(\theta(z) = 0\). If there is a uniform tilt, then we can simply reduce \(\Delta n\) for all the birefringent plates.
The twist angle is given by

\[ \phi(z) = qz = \Phi \frac{z}{d} \]

where \( \Phi \) is the total twist angle of the LC cell. All twist angles are measured relative to the x-axis. The above equation already assumes that \( \phi(0) = 0 \), i.e. the input director is parallel to the x-axis.

Assuming N plates, then there are N Jones matrices. The twist angle of the nth plate is

\[ \phi_n = n \Delta \phi \]

where \( \Delta \phi = \frac{\Phi}{N} \)

Therefore the LC Jones matrix is given by

\[ M_{LC} = (R_{N\Delta \phi}^{-1}M'R_{N\Delta \phi})\cdots(R_{2\Delta \phi}^{-1}M'R_{2\Delta \phi})(R_{\Delta \phi}^{-1}M'R_{\Delta \phi}) \]

where

\[
M' = \begin{pmatrix}
  e^{-j\delta/N} & 0 \\
  0 & e^{j\delta/N}
\end{pmatrix}
\]
with \( \delta = \frac{\pi d \Delta n}{\lambda} = d\Delta k \)

where \( \Delta k = \frac{k_e - k_o}{2} = \frac{\pi \Delta n}{\lambda} \)

Now \( R_{n\Delta \phi} R_{(n-1)\Delta \phi}^{-1} = R_{\Delta \phi} \)

So \( M_{LC} = R_{\Phi}^{-1} (M'R_{\Delta \phi})^N \)

We can now make use of the Chebychev identity to simplify this matrix further. It can be shown that

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^N = \begin{pmatrix} AU_N - U_{N-1} & BU_N \\ CU_N & DU_N - U_{N-1} \end{pmatrix}
\]

where \( U_N = \frac{\sin N\Omega}{\sin \Omega} \)

and \( \cos \Omega = \frac{1}{2} (A + D) \)

This is valid for any unitary matrix with \( AD - BC = 1 \).

(Exercise: Prove the Chebychev identity by induction)

Then it is straight-forward to show that as \( N \to \infty \),
\[(M'R_{\Delta \phi})^N = \begin{pmatrix} \cos \beta d - i \frac{\Delta k}{\beta} \sin \beta d & \frac{\Phi}{\beta d} \sin \beta d \\ -\frac{\Phi}{\beta d} \sin \beta d & \cos \beta d + i \frac{\Delta k}{\beta} \sin \beta d \end{pmatrix}\]

where \( \beta^2 d^2 = \Phi^2 + \Delta k^2 d^2 \)

Finally, \( M_{LC} \) can be written as a simple matrix

\[M_{LC} = \begin{pmatrix} a - ib & c - id \\ c - id & a + ib \end{pmatrix}\]

where
\[
\begin{align*}
    a &= \cos \Phi \cos \beta d + \frac{\Phi}{\beta d} \sin \Phi \sin \beta d \\
    b &= \frac{\Delta k}{\beta} \cos \Phi \sin \beta d \\
    c &= \sin \Phi \cos \beta d - \frac{\Phi}{\beta d} \cos \Phi \sin \beta d \\
    d &= \frac{\Delta k}{\beta} \sin \Phi \sin \beta d
\end{align*}
\]

This is an extremely useful result.

**Properties:**
1. \( M_{LC} \) is normalized, i.e. \( |M_{LC}| = 1 \). (Check it.)
(2) $M_{LC}$ is unitary, i.e. $M_{LC}^*T = M_{LC}^{-1}$. (Check it). Recall from matrix algebra: the eigenvalues of unitary matrices have forms $e^{i\alpha}$ (or unit length).

(3) $M_{LC}$ changes if we change the twist sense, i.e. $\Phi \rightarrow -\Phi$. The off-diagonal elements changes sign. But the properties remains the same. (e.g. the eigenvalues and eigenvectors are the same.) Therefore it does not matter how we define the twist sense, as long as it is consistent.

(4) If the wave propagates in the opposite direction, i.e. we have a left-handed coordinate system, then $M_{LC}$ becomes $M_{LC}^*$. Proof: if $z \rightarrow -z$, then $M' \rightarrow M'^*$. This is a useful result for reflective displays.

(5) It can be shown that

$$M_{LC} = R_{\Phi}^{-1} T^{-1} \begin{pmatrix} e^{-i\beta d} & 0 \\ 0 & e^{i\beta d} \end{pmatrix} T$$

where $T = \begin{pmatrix} \cos \chi & -i \sin \chi \\ -i \sin \chi & \cos \chi \end{pmatrix}$ and $\sin 2\chi = \phi/\beta d$.

In this form, the LC cell behaves as a “rotating waveplate”.

7.5 Eigenvalues and eigenvectors of $M_{LC}$

The eigenvalues of the LC Jones matrix can be obtained indirectly. Let us first find the eigenvectors of $(M'R_{\Delta\phi})^N$. Recall that $M_{LC} = R_{\Phi}^{-1}(M'R_{\Delta\phi})^N$.

Write $(M'R_{\Delta\phi})^N = \begin{pmatrix} g & h \\ -h & g \end{pmatrix}$
where \( g = \cos \beta d + i \frac{\Delta k}{\beta} \sin \beta d \)

and \( h = \frac{\Phi}{\beta d} \sin \beta d \)

Then the eigenvalues are given by the secular equation:

\[
\begin{bmatrix}
g * -\lambda & h \\
- h & g - \lambda
\end{bmatrix} = 0
\]

The solution is easily derived to be:

\( \lambda = e^{-j\beta d} \) and \( e^{j\beta d} \)

The corresponding **normalized** eigenvectors are

\[ v_1 = \begin{pmatrix}
\sqrt{\frac{\beta + \Delta k}{2\beta}} \\
- j \sqrt{\frac{\beta - \Delta k}{2\beta}}
\end{pmatrix} \]

and

\[ v_2 = \begin{pmatrix}
\sqrt{\frac{\beta - \Delta k}{2\beta}} \\
 j \sqrt{\frac{\beta + \Delta k}{2\beta}}
\end{pmatrix} \]

These are elliptically polarized waves. Note that \( v_1 \cdot v_2 = 1 \) which is another property of unitary matrices.

This result is easily obtained if we note that
\[(M'R_{\Delta \phi})^N = T^{-1} \begin{pmatrix} e^{-i\beta d} & 0 \\ 0 & e^{i\beta d} \end{pmatrix} T\]

where \[T = \begin{pmatrix} \cos \chi & -i \sin \chi \\ -i \sin \chi & \cos \chi \end{pmatrix}\] and \[\sin 2\chi = \phi / \beta d.\]

By inspection we know that the eigenvalues are \(e^{-i\beta d}\) and \(e^{i\beta d}\) and the eigenvectors are

\[
\begin{pmatrix} \cos \chi \\ -i \sin \chi \end{pmatrix} \text{ and } \begin{pmatrix} \sin \chi \\ i \sin \chi \end{pmatrix}
\]

To visualize what is going on inside the LC cell, just take \(d = z\) in all the above formulas, and replace \(\Phi\) by \(\phi(z)\). Then everything else is still valid.

Now \[M_{\text{LC}} = R_{\phi}^{-1} (M'R_{\Delta \phi})^N\]

So \[M_{\text{LC}} v_I = R_{\phi}^{-1} (M'R_{\Delta \phi})^N v_I = \lambda_I R_{\phi}^{-1} v_I\]

Hence inside the LC, the eigenvectors of polarization are rotating elliptically polarized waves. The rotation is in the same sense and pitch as the director twist angle. The physical picture is clearer if we take at the 2 limits of high pitch and low pitch.

(1) **Low twist large birefringence limit** (\(d\Delta k >> \phi\))

In this case \[\beta \sim \Delta k\]

The 2 eigenvectors are
These are linearly polarized light along the x-axis (e-wave) and along the y-axis (o-wave). The waves inside the LC cell are linearly polarized light rotating along as the director. This is called the **waveguiding** limit. It is also called the adiabatic limit or the **Mauguin** limit.

\[ \mathbf{v}_1 \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ \mathbf{v}_2 \sim j \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

(2) **High twist limit small birefringence limit** \((\phi \gg \Delta \kappa)\)

In this case \(\beta \sim \phi/d\)

The 2 eigenvectors are

\[ \mathbf{v}_1 \sim \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -j \end{pmatrix} \]

\[ \mathbf{v}_2 \sim \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ j \end{pmatrix} \]

These are circularly polarized waves. Hence the waves inside the LC cell are rotating circularly polarized waves. It turns out from more rigorous wave propagation theory that only the circularly polarized wave with the same twist sense as the director can propagate. Hence the other circularly polarized wave will be reflected. This is the principle of the cholesteric display.
7.6 Parameter space

All the operating modes of a LCD can be shown on the parameter space.

If the LCD is composed of a polarizer at angle $\alpha$ to the x-axis, an LC cell with input director along the x-axis, and an output polarizer at angle $\gamma$ to the x-axis, then the transmission is given by

$$T = T(\alpha, \gamma, \phi, d\Delta n) = \left| \begin{pmatrix} \cos \gamma & \sin \gamma \end{pmatrix} \cdot M_{LC} \cdot \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \right|^2$$

There are only 4 parameters, in the zero volt state. If we fix any 2 of the parameters, $T$ can be plotted as a function of the other 2 parameters in a 2D contour map. This is called the parameter space. In most cases, the polarizers are either parallel or perpendicular. So $\gamma = \alpha$ or $\gamma = \alpha + \pi/2$. So in fact there are 3 parameters in $T(\alpha, \phi, d\Delta n)$.

Common situation: $\alpha = 0$, $\gamma = 90^\circ$. This is for example the case for a TN display. The following parameter space can be obtained:
Each line represents a constant transmittance contour. The increment is 10% transmittance. The shaded part represents $T > 90\%$. The series of peaks show the Mauguin modes. The series at $90^\circ$ twist is the normal TN display. The series at $270^\circ$ shows the STN display. The series at $180^\circ$ does not show near 100% transmittance. It is called the OMI mode. It has a maximum transmittance of 41%, but has other advantages such as B/W operation and low dispersion.

This parameter space contains a lot of information. It is also very easy to understand the similarities and differences of TN, HTN (High TN), STN (Supertwisted nematic), OMI (Optical mode interference), SBE (Supertwist birefringent effect) and ECB modes.

Nomenclature:
The PS gives the transmittance at no voltage (nonselect). When a voltage is applied, the transmittance will change because $\Delta n$ decreases. For a first approximation, we can regard this change as a vertical line going towards the x-axis.

Also the dispersion can be visualized easily. A change in $\lambda$ is equivalent to change in $\Delta n$ since the parameter $\Delta n/\lambda$ appears together in all formulas. Therefore it is equivalent to a vertical axis scaling.

The parameter space above is for the waveguiding situation, with $\alpha = 0$. If $\alpha = 45^\circ$, the ECB modes will be obtained. Here we show a series of parameter spaces to show the systematic variation of the operating modes of any LCD.
The parameter space can also be plotted with $\alpha$ and $d\Delta n$ as the free parameters. They are useful for designing new LCD operating modes. For example, the following PS shows the 240° twist STN display with cross polarizers. The optimum $d\Delta n$ of 0.75$\mu$m, and optimum polarizer angle of 30° can be obtained easily. This is in agreement with the best design.

7.7 Gooch and Tarry formulas:

Gooch and Tarry derived the analytical expressions of the optical properties of the GTN cell. We can derive those important formulas using the Jones matrix easily.
Case 1: $\alpha = 0, \gamma = \Phi + \pi/2$.

The transmission of the LCD is given by

$$ T = \left| \begin{pmatrix} \cos(\Phi + \pi/2) & \sin(\Phi + \pi/2) \end{pmatrix} \bullet M_{LC} \right| \begin{pmatrix} 1 \\ 0 \end{pmatrix}^2 $$

Therefore

$$ T = \frac{\Phi^2}{\beta^2 d^2} \sin^2 \beta d = \frac{1}{1 + u^2} \sin^2 \beta d $$

where $u = \frac{\delta}{\Phi} = \frac{\pi d \Delta n}{\lambda \Phi} = \frac{d \Delta k}{\Phi}$

This is the original Gooch and Tarry formula. In particular, for the $90^o$ TN cell, the LCD will have parallel polarizers and the transmission is given by

$$ T = \frac{\Phi^2}{\beta^2 d^2} \sin^2 \beta d $$

which can be used to derive the waveguiding modes of the TN LCD. This will be discussed in the next Chapter.

Case 2: $\alpha = 0, \gamma = \Phi$.

The transmission of the LCD is given by

$$ T = \left| \begin{pmatrix} \cos \Phi & \sin \Phi \end{pmatrix} \bullet M_{LC} \right| \begin{pmatrix} 1 \\ 0 \end{pmatrix}^2 $$
Therefore

\[
T = \frac{1}{1+u^2} \{u^2 + \cos^2 \beta d\}
\]

This is just 1-T of case (1).

Here is a plot for \(\phi = 90^\circ, 180^\circ\) and \(270^\circ\). These are the waveguiding modes.
The 90° TN cell with cross polarizers is a special example of this case. Here the transmission is given by

\[ T = 1 - \frac{\Phi^2}{\beta^2 d^2} \sin^2 \beta d \]

**Case 3: \( \alpha = 45^\circ, \gamma = -45^\circ \).**

The transmission of the LCD is given by

\[ T = \frac{1}{4} \left| \begin{pmatrix} 1 & -1 \end{pmatrix} \cdot M_{LC} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right|^2 = b^2 + c^2 \]

This can be used to analyze the ECB mode displays, which will be presented in the next Chapter. Here is a plot of the transmission as a function of \( d \Delta n \) for \( \phi = 0^\circ, 90^\circ, 180^\circ, 270^\circ \). It can be seen that these are truly interference modes.
Case 4: $\alpha = 0, \gamma = \pi/2$

Here the polarizers are always crossed. The transmission is given by

$$T = \frac{1}{4} \begin{vmatrix} 0 & 1 \end{vmatrix} M_{LC} \begin{pmatrix} 1 \end{pmatrix}^2 = c^2 + d^2$$

These examples show the power of the Jones matrix and the parameter space approach in analyzing LCD optics.

### 7.7 Wave propagation theory of LC cell

There is another approach to LCD optics – wave propagation theory. It yields the same results as the Jones matrix approach.

We first obtain the differential equation for the Jones vector.

Let $n = (\cos \phi, \sin \phi, 0)$

$\phi = qz$

$k_m = (n_e + n_o) \pi/\lambda$
\[ \phi_m = k_m \Delta z \]
\[ \Delta k = \pi \Delta n / 2\lambda \]
\[ \delta = \Delta k \Delta z \]

then as before the Jones matrix of the nth slice of LC cell can be approximated by a birefringent plate:

\[ L(z) = e^{j\phi_m} R_{\phi}^{-1} \begin{pmatrix} e^{-j\delta} & 0 \\ 0 & e^{j\delta} \end{pmatrix} R_{\phi} \]

Represent the Jones vector at \( z \) by \( J(z) \), then

\[ J(z + \Delta z) = L(z) J(z) \]

Therefore

\[ \Delta J = J(z + \Delta z) - J(z) = [L(z) - I] J(z) \]

As \( \Delta z \to 0 \),

\[ L(z) \approx (1 - j\phi_m) R_{\phi}^{-1} \begin{pmatrix} 1 - j\delta & 0 \\ 0 & 1 + j\delta \end{pmatrix} R_{\phi} \]

Simplifying, we get

\[ L(z) = (1 - jk_m \Delta z) \left[ I - R_{\phi}^{-1} \begin{pmatrix} j\Delta k & 0 \\ 0 & -j\Delta k \end{pmatrix} R_{\phi} \Delta z \right] \]

Hence the differential equation for \( J(z) \) is given by
\[
\frac{dJ(z)}{dz} = \left[ -j k m - R_\phi^{-1} \begin{pmatrix} j \Delta k & 0 \\ 0 & -j \Delta k \end{pmatrix} R_\phi \right] J(z)
\]

Now we need to get rid of \(R_\phi\) by making the transformation

Let \(J_R(z) = e^{j km z} R_\phi J(z)\)

Then it can be shown that, after a few steps,

\[
\frac{dJ_R(z)}{dz} = j \begin{pmatrix} -\Delta k & -jq \\ jq & \Delta k \end{pmatrix} J_R(z)
\]

Now we need to diagonalize this equation. Let the eigenvalue be \(\beta\), then

\[
\begin{vmatrix}
-\Delta k - \beta & -jq \\
jq & \Delta k - \beta
\end{vmatrix} = 0
\]

which gives

\[
\beta = \pm \sqrt{\Delta k^2 + q^2}
\]

which is the same as \(\beta^2 d^2 = \Phi^2 + \Delta k^2 d^2\) obtained previously in section 7.4.

Now the normalized eigenvectors corresponding to the + and – solutions are
\[ v^+ = \begin{pmatrix} \sqrt{\frac{\beta - \Delta k}{2\beta}} \\ \frac{\beta + \Delta k}{2\beta} \\ j \sqrt{\frac{\beta + \Delta k}{2\beta}} \end{pmatrix} \]

and
\[ v^- = \begin{pmatrix} \sqrt{\frac{\beta + \Delta k}{2\beta}} \\ \frac{\beta - \Delta k}{2\beta} \\ -j \sqrt{\frac{\beta - \Delta k}{2\beta}} \end{pmatrix} \]

Therefore the solutions for \( J_R(z) \) are
\[
J_R^+(z) = v^+ e^{j\beta z} \\
J_R^-(z) = v^- e^{j\beta z}
\]

[Verify that these are indeed the solutions of the differential equation for \( J_R(z) \)].

Therefore the Jones vectors inside the LC cell is given by
\[
J^+(z) = e^{-j(k_m - \beta)z} R_\phi^{-1} v^+ \\
\] and
\[
J^-(z) = e^{-j(k_m + \beta)z} R_\phi^{-1} v^-
\]

These are exactly the same results as in section 7.5. The waves inside the LC cell are elliptically polarized waves with rotating axis guided by the LC director. The results here also provide the additional phase factors.
The wave propagation approach is especially useful for cholesteric case if we take into consideration continuous reflection inside the cell. It will give the reflection of the circular polarized light, which we cannot get here.
Chapter 8. LCD Optical Modes

LCD can operate in many modes. Every point in the parameter space can be a quiescent operating point for the LCD. Depending on the polarizer angle, an LCD can be in either the ECB, or waveguiding or mixed mode of operation.

8.1 ECB modes (Interference modes)

Classic ECB: No twist

There are several classic ECB modes (homogeneous cell, hybrid aligned cell, homeotropic cell). They all rely on the birefringence of the LC cell. The LC cell behaves as a retardation plate with variable retardation.

The polarizer and the input director are always at 45° to each other. The polarizers can be cross (as shown) or parallel.

For a cross polarizer geometry, the transmission is given by
\[ T = \sin^2 \int_0^d \frac{\pi \Delta n(z)}{\lambda} \, dz \]

where \( d = \) film thickness
\( \Delta n = \) birefringence
\( \lambda = \) wavelength

For a parallel-parallel polarizer geometry

\[ T = 1 - \sin^2 \int_0^d \frac{\pi \Delta n(z)}{\lambda} \, dz \]

Example: for a homogeneous cell, \( \Delta n(z) = \) constant
As the retardation depends on $V$, so the transmission will also depend on $V$. The shape of the curve depends on the initial retardation value. For example, $d\Delta n(0)=2$ $\mu$m:

![Graph showing the transmission of ECB cell vs. voltage]

This was calculated using DIMOS, and is an example of an electro-optic curve.

**General ECB modes:**

The general ECB mode can be analyzed using the parameter space. For a LC cell with any twist, the interference mode or ECB mode can be obtained by placing the input director at $45^\circ$ to the input director. The output polarizer is at $90^\circ$ to the input polarizer. In this case, it can be shown (homework exercise) easily that

$$T = b^2 + c^2$$
In particular, if $\Phi = 0$, it can be easily shown that

$$T = \sin^2 \delta$$

which is what we have derived for a single birefringent plate with no twist. So the above formula is just an extension of the ECB mode to twisted nematic cells.

It should noted that the STN and SBE modes with near $45^\circ$ polarizer angles, are actually general ECB modes. They are not waveguiding and are dispersive (colored).

The following curves show examples of ECB modes for $90^\circ$ twist and $180^\circ$ twist cells.

For the interference modes, the transmission is always periodic in $d\Delta n$. There is no waveguiding limit.
For ECB mode, there can be several maxima and several minima. If the initial retardation value is reduced, it can have just one peak.

**Dispersion:**

Because of the dependence of $T$ on wavelength, ECB displays are intrinsically dispersive, meaning that there is strong coloring of the transmitted light. This is bad for many applications but is good for some applications requiring color contrast.

For classic ECB, the spectrum can be calculated easily. If $\Delta n = \text{constant independent of } z$, then the peaks corresponds to

\[
\frac{d\Delta n}{\lambda_o} = M + \frac{1}{2}
\]

for $M = 1, 2, 3, \ldots$

where $\lambda_o$ is the peak wavelength.

So

\[
T = \sin^2 \left( M - \frac{1}{2} \right) \frac{\pi \lambda_o}{\lambda}
\]

for the ON states

Similarly,

\[
T = \sin^2 M \frac{\pi \lambda_o}{\lambda}
\]

for the OFF state

Note that the ECB cell is designed to be either normally on or normally off, but not both, obviously.
An ECB display does not have a true dark state. At any V, there is some $\Delta n$, and that must correspond to peak transmission of some color. So the color of the display changes as V changes.

In the ECB mode, the input polarizer is always at 45° to the input director.

(Homework) Calculate the spectrum for the general ECB mode (i.e. with a twist).

8.2 Waveguiding (Mauguin) modes

In the waveguiding mode, the input polarizer is at 0° or 90° to the input director of the LC cell. In this case, the polarization of the light rotates in the same way as the LC director of the LC cell. Therefore if the LC director twist is
90°, the polarization also twist by 90°. This rotation is supposedly independent of wavelength.

Typical configuration of a TN LCD:

Let us apply the LC Jones matrix to analyze the waveguiding modes. Here the input polarizer is parallel to the input director and the output polarizer is parallel to the output director.
We have shown above that the transmission is given by the famous Gooch and Tarry formula.

\[ T = \frac{1}{1+u^2} \{u^2 + \cos^2 \beta d\} \]

Let us examine the case of a 90° twist TN cell. In this case, the transmission is given by

\[ T = 1 - \frac{\Phi^2}{\beta^2 d^2} \sin^2 \beta d \]

where \( \beta d = \sqrt{\left(\frac{\pi}{2}\right)^2 + \left(\frac{\pi d \Delta n}{\lambda}\right)^2} \)

A plot of \( T \) vs \( \frac{d \Delta n}{\lambda} \) shows several peaks.
The transmission is 100% when

\[ \beta d = N\pi \quad \text{for } N = 1, 2, 3\ldots \]

So the transmission peaks are at

\[ \frac{d\Delta n}{\lambda} = \frac{1}{2} \sqrt{4N^2 - 1} \]

These are called the Mauguin minima. (It will be minimum transmission if the polarizers are parallel). In the waveguiding limit, the transmission is 100% always, because the solution is a rotating linearly polarized wave. It occurs at large \(d\Delta n\). For finite \(d\Delta n\), the waves are slightly elliptical. The ellipticity parameter is defined as the ratio of the major to the minor axis.

**Definitions:** If \( \mathbf{P}_{\text{in}} \) is parallel to \( \mathbf{D}_{\text{in}} \), the wave inside the cell is an e-wave. This is called an **e-mode** TN cell. If \( \mathbf{P}_{\text{in}} \perp \mathbf{D}_{\text{in}} \), then the wave inside the cell is an o-wave. This is **o-mode** operation. Whether an o-mode or an e-mode is used depends also on the viewing angle requirements.

Most TN LCDs operate either in the first or second minimum. Here are their values

<table>
<thead>
<tr>
<th></th>
<th>(d\Delta n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>First minimum</td>
<td>0.475 (\mu\text{m})</td>
</tr>
<tr>
<td>Second minimum</td>
<td>1.075 (\mu\text{m})</td>
</tr>
</tbody>
</table>
Recall from the discussion on refractive index of LC that most LC have $\Delta n$ of 0.07-0.2. Therefore one can choose the right combination of cell gap and $\Delta n$ to make first or second minimum cells.

Sometimes, the choice of the first or second minimum also has to do with viewing angle.

We shall show that the description of waveguiding modes for the Mauguin minima is correct only at large $d$ – the waveguiding limit. There is some degree of birefringence in the normal first minimum or second minimum operation for TN LCD.

Dispersion: the waveguiding effect is $\lambda$ independent. Therefore the waveguiding modes are true black and white – no dispersion.

### 8.3. Mixed modes

If the polarizer angle is at angles other than 0, $45^\circ$, or $90^\circ$ to the input director of the LC cell, then we have a mixed mode situation. The LCD operates somewhere between the ECB and the waveguiding limit.

In transmittive LCD, mixed modes are not used. The optimized STN may have a little bit of mixed mode behavior because the polarizer and analyzer angles may not be $45^\circ$ to the directors exactly, and also they are not parallel or perpendicular to each other.
8.4 Reflective modes: MTB (mixed TN and birefringence mode)

A reflective display is different from a transflective display. There is only one polarizer.

For such a truly reflective LCD without the rear polarizer, one has to use the MTB mode or the ECB mode. The analysis is greatly simplified by use of the Jones matrix.

For a reflective LCD with only one polarizer, the reflectance is given by

\[
R = \begin{vmatrix} \cos \alpha & \sin \alpha \end{vmatrix} \cdot R_\Phi M_{LC}^{-1} M_{LC} \cdot \begin{vmatrix} \cos \alpha \\ \sin \alpha \end{vmatrix}^2
\]

where \( M^* \) means opposite twist sense for the LC.

The following figures are parameter space for this display with various polarizer angles.
The minima in reflectance are called the TN-ECB modes, the MTN modes for $\alpha = 0$. For nonzero $\alpha$ they are called MTB modes.

The MTB modes are newly discovered by HKUST. There are many new uses of such modes for low power PDA and fancy applications.
There are a lot of interesting physics contained in these parameter space diagrams. For the case of $\alpha = 45^\circ$, the zero twist modes are exactly the ECB modes. For the case of $\alpha = 0$, the TN-ECB modes can be derived exactly.

Given a $x$-polarized input wave, the ellipticity of the output through the LC cell is given by the Gooch Tarry formula:

$$\chi = \tan \left[ \frac{1}{2} \sin^{-1} \left( \frac{2u}{1+u^2} \sin^2 \Phi \sqrt{1+u^2} \right) \right]$$

Therefore, for the TN-ECB mode, the reflected wave should be cross polarized. i.e. the LC cell should behave as a quarterwave plate. For this to happen, $\chi$ has to be 1.

Then it can be derived easily (homework exercise) that this leads to the solution

$$\Phi = (2N - 1) \frac{\pi}{2\sqrt{2}} \quad \text{where } N=1,2,3...$$

and the corresponding retardation is given by

$$d\Delta n = \lambda \Phi/\pi$$

For example, at $\lambda=550\text{nm}$, the first 2 TN-ECB modes are $(63^\circ, 0.18\mu\text{m})$, $(189^\circ, 0.54\mu\text{m})$. They corresponds exactly to the solutions depicted in the PS.