Exact controllability of semilinear evolution equation and applications

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Abstract: In this paper we characterise the exact controllability for the following semilinear evolution equation
\[ z' = Az + Bu(t) + F(t, z, u(t)), \quad t > 0, \quad z \in Z, \quad u \in U, \]
where \( Z, U \) are Hilbert spaces, \( A : D(A) \subset Z \rightarrow Z \) is the infinitesimal generator of strongly continuous semigroup \( \{T(t)\}_{t \geq 0} \) in \( Z \), \( B \in L(U, Z) \), the control function \( u \) belongs to \( L^2(0, \tau; U) \) and \( F : [0, \tau] \times Z \times U \rightarrow Z \) is a suitable function. First, we give a necessary and sufficient condition for the exact controllability of the linear system \( z' = Az + Bu(t) \); Second, under some conditions on \( F \), we prove that the exact controllability of the linear system is preserved by the semilinear system, in this case the control \( u \) steering an initial state \( z_0 \) to a final state \( z_1 \) at time \( \tau > 0 \) is given by the following formula: \( u(t) = B^* T^*(\tau - t) W^{-1}(I + K)^{-1}(z_1 - T(\tau)z_0), \) according to Theorem 3.1. Finally, these results can be applied to the controlled damped wave equation.

Keywords: semilinear evolution equation; controllability; damped wave equation.


1 Introduction

In this paper we characterise the exact controllability for the following semilinear evolution equation

\[ z' = Az + Bu(t) + F(t, z, u(t)), \quad z \in Z, \quad u \in U, \quad t > 0, \quad (1.1) \]

where \( Z, U \) are Hilbert spaces, \( A : D(A) \subset Z \rightarrow Z \) is the infinitesimal generator of strongly continuous semigroup \( \{T(t)\}_{t \geq 0} \) in \( Z, B \in L(U, Z) \), the control function \( u \) belongs to \( L^2(0, \tau; U) \) and \( F : [0, \tau] \times Z \times U \rightarrow Z \) is a suitable function. First, we give a necessary an sufficient condition for the exact controllability of the linear system

\[ z' = Az + Bu(t), \quad z \in Z, \quad u \in U, t > 0. \quad (1.2) \]

In this case, the control \( u \in L^2(0, \tau; U) \) steering an initial state \( z_0 \) to a final state \( z_1 \) at time \( \tau > 0 \) is given by the following formula:

\[ u(t) = B^*T^*(\tau - t)W^{-1}(z_1 - T(\tau)z_0), \]

where \( W : Z \rightarrow Z \) is the linear operator

\[ Wz = \int_0^\tau T(\tau - s)BB^*T^*(\tau - s)zds, \quad \forall z \in Z; \]

according to Theorem 4.1.

Second, under some conditions on \( F \), we prove that the exact controllability of the linear system (1.2) is preserved by the semilinear system (1.1); in this case, the control \( u \in L^2(0, \tau; U) \) steering an initial state \( z_0 \) to a final state \( z_1 \) at time \( \tau > 0 \) (using the non linear system (1.1)) is given by the following formula:

\[ u(t) = B^*T^*(\tau - t)W^{-1}(I + K)^{-1}(z_1 - T(\tau)z_0), \]

where \( K : Z \rightarrow Z \) is the operator given by

\[ K\xi = \int_0^\tau T(\tau - s)F(s, z_\xi(s), (S\xi)(s))ds, \]

and \( z_\xi(\cdot) \) is the solution of equation (1.1) corresponding to the control \( u \) define by

\[ u(t) = (S\xi)(t) = B^*T^*(\tau - t)W^{-1}\xi, \quad t \in [0, \tau]; \]

according to Theorem 3.1.

Finally, we apply these results to the following controlled damped wave

\[ \begin{align*}
    w_{tt} + cw_t - dw_{xx} &= u(t, x) + f(t, u(t, x), w, w_t), \quad 0 < x < 1 \\
    w(t, 0) &= w(t, 1) = 0, \quad t \in \mathbb{R}
\end{align*} \quad (1.3) \]
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where \( d > 0, c \geq 0 \), the distributed control \( u \in L^2(0, t_1; L^2(0, 1)) \) and the nonlinear term \( f(t, w, v, u) \) is a Lipschitz function \( f : [0, t_1] \times \mathbb{R}^3 \to \mathbb{R} \), i.e., there exists a constant \( L > 0 \) such that, for all \((t, w_1, v_1, u_1), (t, w_2, v_2, u_2) \in [0, t_1] \times \mathbb{R}^3 \) we have:

\[
|f(t, w_2, v_2, u_2) - f(t, w_1, v_1, u_1)| \leq L\{|w_2 - w_1| + |v_2 - v_1| + |u_2 - u_1|\}. \tag{1.4}
\]

Moreover, we compute the control steering the initial state \((w(0), w_t(0))\) to the final state \((w(t_1), w_t(t_1))\).

The exact controllability is a hard problem even for linear system, for that reason there are more results on approximate controllability of system like (1.2). For exact controllability of semilinear equation one can see, for example, the papers by Liu and Williams (1997), Zhang (1999, 2000, 2001) and Zuazua (1990, 1993). Nevertheless, the non linear terms here depends on all variables appearing in the equation, including the control. Finally, our technique is very general and can be applied to many systems of partial differential equation.

We shall apply the following well known Theorem used to characterise centre manifolds in dynamical system theory:

**Theorem 1.1:** Let \( Z \) be a Banach space and \( K : Z \to Z \) a Lipschitz function with a Lipschitz constant \( k < 1 \) and consider \( G(z) = z + Kz \). Then \( G \) is an homeomorphis whose inverse is a Lipschitz function with a Lipschitz constant \((1 - k)^{-1}\).

The outline of the paper is as follows:

In Section 2, we present some results on exact controllability for the linear system (1.2) and prove some new one. In Section 3, we present and prove the main results of this work. In Section 4, we apply these results to the controlled damped wave equation (4.1). Finally, in Section 5, the conclusion section, we point out that our result can be applied to a broad class of second order diffusion equation in Hilbert spaces.

### 2 Controllability of the linear system

In this section we shall presents some results on the controllability of the linear system (1.2) that we will use in the next section. To this end, for all \( z_0 \in Z \) and \( u \in L^2(0, \tau; U) \) the the initial value problem

\[
\begin{cases}
z' = Az(t) + Bu(t), & t > 0, \\
z(0) = z_0,
\end{cases}
\tag{2.1}
\]

admits only one mild solution given by

\[
z(t) = T(t)z_0 + \int_0^t T(t - s)Bu(s)ds, & t \in [0, \tau].
\tag{2.2}
\]

**Definition 2.1 (Exact Controllability):** The system (1.2) is sad to be exactly controllable on \([0, \tau]\) if for all \( z_0, z_1 \in Z \) there exists a control \( u \in L^2(0, \tau; U) \) such that the solution \( z(t) \) of (2.2), corresponding to \( u \), verifies: \( z(\tau) = z_1 \).
Consider the following bounded linear operator:

\[ G : L^2(0, \tau; U) \rightarrow Z, \quad Gu = \int_0^\tau T(\tau - s)Bu(s)ds, \quad (2.3) \]

whose adjoint operator \( G^* : Z \rightarrow L^2(0, \tau; U) \) is given by

\[ (G^* \xi)(s) = B^*T^*(\tau - s)\xi. \quad (2.4) \]

Then, if we define the operators \( W = GG^* : Z \rightarrow Z \) we obtain that

\[ Wz = \int_0^\tau T(\tau - s)BB^*T^*(\tau - s)z ds. \quad (2.5) \]

The following Theorem from Curtain and Zwart (1995) is a characterisation of the exact controllability of the linear system (1.2).

**Theorem 2.1:** For the system (1.2) we have the following condition for exact controllability.

System (1.2) is exactly controllable on \([0, \tau]\) if, and only if, any one of the following condition hold for some \( \gamma > 0 \) and all \( z \in Z \):

(i) \( G(L^2(0, \tau; U)) = \text{Range}(G) = Z \),

(ii) \( \langle Wz, z \rangle \geq \gamma \|z\|^2 \),

(iii) \( \|G^*z\|^2 := \int_0^\tau \| (Gz^*)(s) \|^2 ds \geq \gamma \|z\|^2 \),

(iv) \( \int_0^\tau \| B^*T^*(\tau - s)z \|^2 \geq \gamma \|z\|^2 \),

(v) \( \text{Ker}(G^*) = \{0\} \) and \( \text{Rang}(G^*) \) is closed.

Now, we are ready to formulate and prove a new result on exact controllability of the linear system (1.2).

**Theorem 2.2:** The system (1.2) is exactly controllable on \([0, \tau]\) if, and only if, the operator \( W \) is invertible. Moreover, the control \( u \in L^2(0, \tau; U) \) steering an initial state \( z_0 \) to a final state \( z_1 \) at time \( \tau > 0 \) is given by the following formula:

\[ u(t) = B^*T^*(\tau - t)W^{-1}(z_1 - T(\tau)z_0). \quad (2.6) \]

**Proof:** Suppose the system (1.2) is exactly controllable on \([0, \tau]\). Then, from the foregoing Theorem we obtain

\[ \gamma^2 \| B^*T^*(\tau - \cdot)z \|^2_U \geq \|z\|^2_Z, \quad z \in Z. \]

i.e.,

\[ \gamma^2 \int_0^\tau \| B^*T^*(\tau - s)z \|^2_U \geq \|z\|^2_Z, \quad z \in Z. \]
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i.e.,
\[ \frac{\gamma^2}{2} \int_0^\tau \langle B^* T^*(\tau - s)z, B^* T^*(\tau - s)z \rangle_{U,U} \geq \|z\|_Z^2, \quad z \in Z. \]

i.e.,
\[ \frac{\gamma^2}{2} \int_0^\tau \langle T(\tau - s)BB^* T^*(\tau - s)z, z \rangle_{U,U} \geq \|z\|_Z^2, \quad z \in Z. \]

Therefore,
\[ \langle Wz, z \rangle \geq \frac{1}{\gamma^2} \|z\|_Z^2, \quad z \in Z. \] 

(2.7)

This implies that \( W \) is one to one. Now, we shall prove that \( W \) is surjective. That is to say
\[ \mathcal{R}(W) = \text{Range}(W) = Z. \]

For the purpose of contradiction, let us assume that \( \mathcal{R}(W) \) is estrictly contained in \( Z \). Using Cauchy Schwarz’s inequality and equation (2.7) we get
\[ \|Wz\| \geq \frac{1}{\gamma^2} \|z\|_Z, \quad z \in Z, \]

which implies that \( \mathcal{R}(W) \) is closed. Then, from Hahn Banach’s Theorem there exists \( z_0 \in Z \) with \( z_0 \neq 0 \) such that
\[ \langle Wz, z_0 \rangle = 0, \quad \forall z \in Z. \]

In particular, putting \( z = z_0 \) we get from equation (2.7) that
\[ 0 = \langle Wz_0, z_0 \rangle \geq \frac{1}{\gamma^2} \|z_0\|_Z^2. \]

Then \( z_0 = 0 \), which is a contradiction. Hence, \( W \) is a bijection and from the open mapping Theorem \( W^{-1} \) is a bounded linear operator.

Now, suppose \( W \) is invertible. Then, given \( z \in Z \) we shall prove the existence of a control \( u \in L^2 \) such that \( Gu = z \). This control \( u \) can be taking as follows
\[ u(t) = B^* T^*(\tau - t)W^{-1}z. \]

In fact,
\[ Gu = \int_0^\tau T(\tau - s)Bu(s)ds = \int_0^\tau T(\tau - s)BB^* T^*(\tau - s)W^{-1}zds = WW^{-1}z = z. \]

In the same way we can prove that the control \( u \) given by equation (2.6) steers the initial state \( z_0 \) to the final state \( z_1 \) in time \( \tau \).

**Corollary 2.1:** If the system (1.2) is exactly controllable, then the operator \( S : Z \rightarrow L^2(0, \tau; U) \) define by
\[ S\xi = G^*W^{-1}\xi \quad \text{or} \quad (S\xi)(s) = B^* T^*(\tau - s)W^{-1}\xi, \]

(2.8)
is a right inverse of \( G \), i.e., \( G \circ S = I \).
3 Controllability of the nonlinear system

We assume that $F$ is good enough such that the equation (1.1) with the initial condition $z(0) = z_0$ and a control $u \in L^2(0, \tau; U)$ admits only one mild solution given by

$$
z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)ds \nonumber + \int_0^t T(t-s)F(s, z(s), u(s))ds \quad t \in [0, \tau]. \tag{3.1}
$$

**Definition 3.1:** The system (1.1) is said to be exactly controllable on $[0, \tau]$ if for all $z_0, z \in Z$, there exists a control $u \in L^2(0, \tau; U)$ such that the corresponding solution $z$ of (3.1) satisfies $z(\tau) = z_1$.

Define the following operator: $G_F : L^2(0, \tau; U) \rightarrow Z$ by

$$
G_F u = \int_0^\tau T(\tau-s)Bu(s)ds + \int_0^\tau T(\tau-s)F(s, z(s), u(s))ds = Gu + \int_0^\tau T(\tau-s)F(s, z(s), u(s))ds, \tag{3.2}
$$

where $z(t) = z(t, u)$ is the solution of equation (3.1) corresponding to the control $u$.

Then, the following proposition is trivial and characterises the exact controllability of equation (1.1).

**Proposition 3.1:** The system (1.1) is exactly controllable on $[0, \tau]$ if, and only if, $\text{Rang}(G_F) = Z$.

So, in order to prove exact controllability of system (1.1) we have to verify the condition of the foregoing proposition. To this end, we need to assume that the linear system (1.2) is exactly controllable. In this case we know from Corollary 2.1 that the operator $S$ defined by equation (2.8) is a right inverse of $G$, so if we put $G_F = G_F \circ S$, we get the operator equation for the exact controllability

$$
\tilde{G}_F \xi = G_F \circ S \xi = \xi + \int_0^\tau T(\tau-s)F(s, z_\xi(s), (S\xi)(s))ds \tag{3.3}
$$

where $z_\xi(\cdot)$ is the solution of equation (3.1) corresponding to the control $u$ defined by

$$
u(t) = (S\xi)(t) = B^*T^*(\tau-t)W^{-1}\xi, \quad t \in [0, \tau].
$$

Hence, if we define the operator $K : Z \rightarrow Z$ by

$$
K \xi = \int_0^\tau T(\tau-s)F(s, z_\xi(s), (S\xi)(s))ds. \tag{3.4}
$$

Equation (3.3) can be written as follows

$$
\tilde{G}_F \xi = \xi + K \xi = (I + K) \xi, \quad \xi \in Z. \tag{3.5}
$$
Now, we shall prove some abstract results making assumptions on the operator $K$. After that, we will put conditions of the nonlinear term $F$ that imply condition on $K$.

**Theorem 3.1:** If the linear system (1.2) exactly controllable on $[0, \tau]$ and the operator $K$ is globally Lipschitz with a Lipschitz constant $k < 1$, then the nonlinear system (1.1) is exactly controllable on $[0, \tau]$ and the control steering the initial state $z_0$ to the final state $z_1$ is given by

$$u(t) = B^*T^*(\tau - t)W^{-1}(I + K)^{-1}(z_1 - T(\tau)z_0).$$

**Proof:** It follows directly from equation (3.3) and Theorem 3.1.

**Theorem 3.2:** If the system (1.2) is exactly controllable on $[0, \tau]$ and the operator $K$ is linear with $K \geq 0$, then the system (1.1) is exactly controllable on $[0, \tau]$ and the control $u(t)$ steering the initial state $z_0$ to the final state $z_1$ is given by

$$u(t) = B^*T^*(\tau - t)W^{-1}(I + K)^{-1}(z_0 - T(\tau)z_1).$$

**Proof:** Clearly that $\tilde{G}_p = I + K$ is one to one and

$$\|\tilde{G}_p z\| > \|z\|, \quad z \in Z,$$

which implies that $\mathcal{R}(\tilde{G}_p)$ is closed. Now, we shall prove that $\tilde{G}_p$ is surjective. That is to say

$$\mathcal{R}(\tilde{G}_p) = \text{Range}(\tilde{G}_p) = Z.$$

For the purpose of contradiction, let us assume that $\mathcal{R}(\tilde{G}_p)$ is strictly contained in $Z$. Then, from Hahn Banach’s Theorem there exists $z_0 \in Z$ with $z_0 \neq 0$ such that

$$\langle \tilde{G}_p z, z_0 \rangle = \langle z + Kz, z_0 \rangle = 0, \quad \forall z \in Z.$$

In particular, putting $z = z_0$ we get that

$$\langle \tilde{G}_p z_0, z_0 \rangle = \|z_0\|^2 + \langle Kz_0, z_0 \rangle = 0.$$

Then $z_0 = 0$, which is a contradiction. Hence, $\tilde{G}_p$ is a bijection and from the open mapping Theorem $\tilde{G}_p^{-1} = (I + K)^{-1}$ is a bounded linear operator. The remainder of the proof follows from here.

The proof of the following lemma follows as in the same way as Lemma 5.1 from Leiva (2005).

**Lemma 3.1:** If $F$ satisfies the Lipschitz condition

$$\|F(t, z_2, u_2) - F(t, z_1, z_1)\| \leq L(\|z_2 - z_1\| + \|u_2 - u_1\|), \quad z_2, z_1 \in Z; u_2, u_1 \in U, t \in [0, \tau],$$
Theorem 3.3: If $F$ satisfies the foregoing Lipschitz condition, the linear system (1.2) is exactly controllable on $[0, \tau]$ and the control steering the initial state $z_0$ to the final state $z_1$ is given by

$$u(t) = B^*T^*(\tau - t)W^{-1}(I + K)^{-1}(z_1 - T(\tau)z_0).$$

Proof: From Lemma 3.1 we know that $K$ is a Lipschitz function with a Lipschitz constant $k$ given by

$$k = (M\|B\| + L)e^{ML\tau} + 1.$$  

and from condition (3.7) we get that $k < 1$. Hence, applying Theorem 3.1 we complete the proof.

4 Application to the damped wave equation

As an application we consider the following controlled damped wave equation

$$\begin{cases}
w_{tt} + cw_t - dw_{xx} = u(t, x) + f(t, u(t, x), w, w_t), & 0 < x < 1 \\
w(t, 0) = w(t, 1) = 0, & t \in \mathbb{R}
\end{cases}$$

where $d > 0$, $c \geq 0$, the distributed control $u \in L^2(0, t_1; L^2(0, 1))$ and the nonlinear term $f(t, w, v, u)$ is a function $f : [0, t_1] \times \mathbb{R}^3 \to \mathbb{R}$.

4.1 Abstract formulation of the problem

Now we will choose the space in which this problem will be set as an abstract second order ordinary differential equation.

Let $X = L^2[0, 1]$ and consider the linear unbounded operator $A : D(A) \subset X \to X$ defined by $A\phi = -\phi_{xx}$, where

$$D(A) = \{\phi \in X : \phi, \phi_x, \text{ are a.e. } \phi_{xx} \in X; \phi(0) = \phi(1) = 0\}.$$
The operator $A$ has the following very well known properties: the spectrum of $A$ consists of only eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty,$$

each one with multiplicity one.

- There exists a complete orthonormal set $\{\phi_n\}$ of eigenvectors of $A$.
- For all $x \in D(A)$ we have
  $$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, \phi_n \rangle \phi_n = \sum_{n=1}^{\infty} \lambda_n E_n x,$$

  where $\langle \cdot, \cdot \rangle$ is the inner product in $X$ and
  $$E_n x = \langle x, \phi_n \rangle \phi_n, \quad \lambda_n = n^2 \pi^2 \quad \text{and} \quad \phi_n(x) = \sin n\pi x.$$

  So, $\{E_n\}$ is a family of complete orthogonal projections in $X$ and
  $$x = \sum_{n=1}^{\infty} E_n x, \quad x \in X.$$

- $-A$ generates an analytic semigroup $\{e^{-At}\}$ given by
  $$e^{-At} x = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n x.$$

The fractional powered spaces $X^r$ are given by:

$$X^r = D(A^r) = \left\{ x \in X : \sum_{n=1}^{\infty} (\lambda_n)^{2r} \|E_n x\|^2 < \infty \right\}, \quad r \geq 0,$$

with the norm

$$\|x\|_r = \|A^r x\|_2 = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 \right\}^{1/2}, \quad x \in X^r,$$

and

$$A^r x = \sum_{n=1}^{\infty} \lambda_n^r E_n x.$$

Also, for $r \geq 0$ we define $Z_r = X^r \times X$, which is a Hilbert Space with norm given by

$$\left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|_{Z_r}^2 = \|u\|^2 + \|v\|^2.$$

Using the change of variables $u' = v$, the second order equation (4.1) can be written as a first order system of ordinary differential equations in the Hilbert space $Z_{1/2} = D(A^{1/2}) \times X = X^{1/2} \times X$ as:

$$z' = Az + Bu + F(t, z, u(t)), \quad z \in Z_{1/2}, \quad t \geq 0,$$

(4.7)
is an unbounded linear operator with domain $D(A) = D(A) \times X$ and

$$F(t, z, u) = \begin{bmatrix} 0 \\ f(t, u, w, v) \end{bmatrix},$$

(4.9)

and the function $F : [0, t_1] \times Z_{1/2} \times X \to Z_{1/2}$. Since $X^{1/2}$ is continuously included in $X$ we obtain for all $z_1, z_2 \in Z_{1/2}$ and $u_1, u_2 \in X$ that

$$\|F(t, z_2, u_2) - F(t, z_1, u_1)\|_{Z_{1/2}} \leq L \{ \|z_2 - z_1\|_{1/2} + \|u_2 - u_1\| \}, \quad t \in [0, t_1].$$

(4.10)

Throughout this section, without loss of generality, we will assume that

$$c^2 < 4d\lambda_1.$$

The following proposition follows from from Leiva (2003, 2005).

**Proposition 4.1:** The operator $A$ given by (4.8), is the infinitesimal generator of strongly continuous group $\{T(t)\}_{t \in \mathbb{R}}$ in $Z_{1/2}$ given by

$$T(t)z = \sum_{n=1}^{\infty} e^{A_n t} P_n z, \quad z \in Z_{1/2},$$

(4.11)

where $\{P_n\}_{n \geq 0}$ is a family of complete orthogonal projections on the Hilbert space $Z_{1/2}$ given by

$$P_n = \text{diag}(E_n, E_n), \quad n \geq 1,$$

(4.12)

and

$$A_n = B_n P_n, \quad B_n = \begin{pmatrix} 0 & 1 \\ -d\lambda_n & -c \end{pmatrix}, \quad n \geq 1.$$

(4.13)

This group decays exponentially to zero. In fact, we have the following estimate

$$\|T(t)\| \leq M(c, d)e^{-\frac{c^2}{2}t}, \quad t \geq 0,$$

(4.14)

where

$$\frac{M(c, d)}{2\sqrt{2}} = \sup_n \left\{ 2 \left[ e + \frac{\sqrt{4d\lambda_n - c^2}}{\sqrt{c^2 - 4d\lambda_n}} \right], (2 + d) \left[ \frac{\sqrt{\lambda_n}}{\sqrt{4d\lambda_n - c^2}} \right] \right\}.$$ 

(4.15)
The proof of the following theorem follows in the same way as the one for Theorem 4.1 from Leiva (2005).

**Theorem 4.1:** The system

\[
\begin{cases}
    z' = Az + Bu z \in Z_{1/2}, & t > 0, \\
    z(0) = z_0.
\end{cases}
\]  \tag{4.16}

is exactly controllable on \([0, \tau]\).

**Theorem 4.2:** If the following estimate holds

\[
(M(c, d)[1 + L]e^{M(c, d)\tau} + 1)(M(c, d)^2\|W^{-1}\|) < 1,
\]  \tag{4.17}

then the system (4.7) is exactly controllable on \([0, \tau]\).

**Proof:** It follows from Theorem 3.3 one we observe that in this case \(\|B\| \leq 1\). \(\square\)

### 5 Conclusion

These results can be applied to the following class of second order diffusion system in Hilbert spaces

\[
\begin{align*}
    w'' + A_0w &= u(t) + f(t, w, u), & t > 0, & w \in W \, u \in U,
\end{align*}
\]

where \(W, U\) are Hilbert spaces, \(A_0 : D(A_0) \subset W \to W\) is an unbounded linear operator in \(W\) with the spectral decomposition:

\[
A_0w = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle \phi_{k,j}, w \rangle \phi_{k,j} = \sum_{j=1}^{\infty} \lambda_j E_j w,
\]

where \(E_j w = \sum_{k=1}^{\gamma_j} \langle \phi_{k,j}, w \rangle \phi_{k,j}\) is a complete orthonormal set of eigenvectors of \(-A_0\) correspondent to the eigenvalues \(\lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty\) with multiplicity \(\gamma_n\) and \(-A_0\) generates a strongly continuous semigroup \(\{T(t)\}_{t \geq 0}\) given by

\[
T(t)w = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j w, \quad w \in W, \quad t \geq 0.
\]

The control \(u \in L^2(0, \tau; U)\), and \(f : [0, \tau] \times W \times U \to W\) is a suitable function.

Examples of this class are the following well known systems of partial differential equations:

**Example 5.1:** The \(nD\) wave equation with Dirichlet boundary conditions

\[
\begin{cases}
    \frac{\partial^2 w}{\partial t^2} - \Delta w = u(t, x) + f(t, u(t, x), w), & t \geq 0, \quad x \in \Omega, \\
    w(t, x) = 0, & t \geq 0, \quad x \in \partial\Omega, \\
    w(0, x) = \phi_0(x), \quad \frac{\partial w}{\partial t}(0, x) = \psi_0(x), & x \in \Omega,
\end{cases}
\]  \tag{5.1}
where $\Omega$ is a sufficiently smooth bounded domain in $\mathbb{R}^N$, $u \in L^2([0,r]; L^2(\Omega))$, $\phi_0, \psi_0 \in L^2(\Omega)$ and $f$ is a suitable function.

**Example 5.2:** The model of Vibrating plate

$$\begin{cases}
\frac{\partial^2 w}{\partial t^2} + \Delta^2 w + u(t,x) + f(t,u(t,x),w) & t \geq 0, \ x \in \Omega, \\
w = \Delta w = 0 & t \geq 0, \ x \in \partial\Omega, \\
w(0,x) = \phi_0(x), \ \ \frac{\partial w}{\partial t}(0,x) = \psi_0(x) & x \in \Omega.
\end{cases}$$

(5.2)

where $\Omega$ is a sufficiently smooth bounded domain in $\mathbb{R}^2$, $u \in L^2([0,r]; L^2(\Omega))$, $\phi_0, \psi_0 \in L^2(\Omega)$ and $f$ is a suitable function.

**References**


