Decentralised coordination of groups of autonomous mobile robots

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Abstract: The paper gives a qualitative analysis of the dynamics of a system consisting of several mobile robots coordinating their motion using simple local nearest neighbour rules. We show that under some assumptions the headings of all robots will be eventually constant. We give an example of a multi-agent system with a cyclic dynamics.

Keywords: decentralised control; multi-robot system; flocking motion; multi-agent coordination; groups of autonomous vehicles.


Biographical notes: Andrey V. Savkin received the MS Degree in 1987 and the PhD Degree in 1991 from The Leningrad University, USSR. Since 2000, he has been a Professor with the School of Electrical Engineering and Telecommunications, The University of New South Wales, Sydney, Australia. His current research interests include robust control and filtering, hybrid dynamical systems, networked control systems, computer-integrated manufacturing, control of mobile robots, computer vision, and application of control and signal processing to biomedical engineering and medicine. He has published four books and numerous journal and conference papers on these topics and served as an Associate Editor for several international journals and conferences.

1 Introduction

The study of decentralised control laws for groups of mobile autonomous robots has emerged as a challenging new research area in recent years (Smith et al., 2001; Fax and Murray, 2002; Ogren et al., 2002; Jadbabaie et al., 2003; Tanner et al., 2003; Moreau, 2003; Ren et al., 2003; Savkin, 2004; Hong et al., 2006; Anderson et al., 2006). Broadly speaking, this problem falls within the domain of decentralised control, but the unique aspect of it is that mobile robots in a group are dynamically decoupled, meaning the motion of one robot does not directly affect any of the other robots.
Researchers in this new emerging area are finding much inspiration from biology, where the problem of animal aggregation is a central one in both ecological and evolutionary theory. Animal aggregations, such as schools of fish, flocks of birds, groups of bees, or swarms of social bacteria, are believed to use simple, local motion coordination rules at the individual level that result in remarkable and complex intelligent behaviour at the group level (see e.g., Shaw, 1962; Okubo, 1986; Flierl et al., 1999).

Vicsek et al. (1995) proposed a simple but interesting discrete-time model of a system of several autonomous agents, e.g., particles, moving in the plane. Each agent’s motion is updated using a local rule based on its own state and the state of its ‘neighbours’. Vicsek’ model can be viewed as a special case of a computer model mimicking animal aggregation proposed in Reynolds (1986) for the computer animation industry. Simulation results of Vicsek et al. (1995) show that in Vicsek’s model all agents might eventually move in the same direction despite the absence of centralised coordination. In the paper Jadbabaie et al. (2003) some modification of the Vicsek’s model was introduced and considered. In Jadbabaie et al. (2003) each agent’s heading is updated as the average of headings of its neighbours whereas in Vicsek et al. (1995) the heading is updated as the heading of the averaged neighbours’ velocity vector. This new rule makes mathematical analysis simpler and allows to apply properties of stochastic matrices. The first results on mathematical analysis of this model were given in Jadbabaie et al. (2003). The main results of Jadbabaie et al. (2003) are sufficient conditions for coordination of the system of agents that are given in terms of a family of graphs characterising all possible neighbour relationships among agents. Some further results were obtained in Jadbabaie (2003) and Savkin (2004). However, these results and methods are not applicable to the original model from Vicsek et al. (1995).

The main result of this paper gives a qualitative analysis of Vicsek’s system under some simplifying assumption. It shows that the set of agents after a finite time will always break into several separated groups and all agents from each of those groups will go in the same direction. Furthermore, we describe conditions under which all agents will eventually move in the same directions for any small enough sampling time. Finally, we give an example of the system that exhibits a cyclic behaviour if one of our assumptions does not hold.

2 Multi-robot system

The system studied in Vicsek et al. (1995) consists of \( n \) autonomous agents, e.g., particles, robots, etc., labelled 1–\( n \). All these agents moving in the plane with the same speed but with different headings. The system operates at discrete times \( t = 0, \delta, 2\delta, \ldots \) where \( \delta > 0 \) is a given sampling period. Let \( M > 0 \) be a given integer, and \( r > 0 \) and \( v > 0 \) be given numbers associated with the system. The dynamics of the agent \( i \) is described by the sequence \( \{x_i(t\delta), y_i(t\delta), \theta_i(t\delta)\} \), where \( x_i(t\delta) \in \mathbb{R}, y_i(t\delta) \in \mathbb{R} \) are the coordinates of the agent in the plane, and \( \theta_i(t\delta) \) is its heading. In this paper, we consider the case of headings taking values in the discrete set

\[
\Theta := \left\{ 0, \frac{a}{M}, \frac{2a}{M}, \ldots, \frac{(M-1)a}{M} \right\}
\]

where \( a \in (0, 2\pi] \) is a given number.
At any time \( t=0, \delta, 2\delta, \ldots \), each agent’s heading is updated using the following “nearest neighbour rule”: Let \( \mathcal{N}_i(t) \) be the set of all agents \( j, j \neq i \) that at time \( t\delta \) belong to the disk with the radius \( r \) centred at \((x_i(t\delta), y_i(t\delta))\):

\[
(x_j(t\delta) - x_i(t\delta))^2 + (y_j(t\delta) - y_i(t\delta))^2 \leq r^2. \tag{2.1}
\]

Following Vicsek et al. (1995), the average heading \( A_i(t\delta) \) is defined as

\[
A_i(t\delta) := \arctan \left( \frac{\sin(\theta_i(t\delta)) + \sum_{j \in \mathcal{N}_i(t)} \sin(\theta_j(t\delta))}{\cos(\theta_i(t\delta)) + \sum_{j \in \mathcal{N}_i(t)} \cos(\theta_j(t\delta))} \right). \tag{2.2}
\]

It is obvious that \( A_i(t\delta) \in [0, \alpha) \). Furthermore, there exists a unique element \( [A_i(t\delta)] \in \Theta \) such that \( [A_i(t\delta)] \leq A_i(t\delta) < [A_i(t\delta)] + \frac{1}{M} \). Now, we define the next heading of the agent \( i \) as

\[
\theta_i((t+1)\delta) := [A_i(t\delta)]. \tag{2.3}
\]

Furthermore, the coordinates of the agent \( i \) are defined as

\[
x_i((t+1)\delta) = x_i(t\delta) + v\delta \cos(\theta_i(t\delta)),
\]

\[
y_i((t+1)\delta) = y_i(t\delta) + v\delta \sin(\theta_i(t\delta)). \tag{2.4}
\]

Note that in the model studied in Vicsek et al. (1995), the headings take values in the continuous set \([0, 2\pi)\). Following Savkin (2004), we adopt assumption on headings from a discrete set that slightly simplifies the model. It should also be pointed out that the heading averaging rule (2.2) is different from the rule from Jadabaie et al. (2003) defined as:

\[
A_i(t) := \frac{1}{1 + |\mathcal{N}_i(t)|} \left( \theta_i(t) + \sum_{j \in \mathcal{N}_i(t)} \theta_j(t) \right). \tag{2.5}
\]

The rule (2.5) defines \( A_i(\cdot) \) as the average of headings whereas (2.2) defines \( A_i(\cdot) \) as the heading of the averaged velocity vector. The rule (2.5) makes analysis simpler and allows to apply properties of stochastic matrices Jadabaie et al. (2003). Also, it was pointed out in Jadabaie et al. (2003), that the use of the averaging rule (2.5) can sometimes have counter-intuitive consequences. For example, by equation (2.5), the average of headings \( 0.001 \) and \( 2\pi - 0.001 \) is \( \pi \) so this might cause two agents with almost zero headings change their headings to almost opposite directions. Note that such a case is impossible if the headings take values in the interval \([0, \pi)\).

**Remark 2.1:** Notice that the system (2.2), (2.3), (2.4) can be viewed as a hybrid dynamical system; see e.g., Savkin et al. (1996), Savkin and Evans (1998, 2002), Matveev and Savkin (2000), van der Schaft and Schumacher (1999) and Savkin and Matveev (2000).
3 The main results

Following Jadbabaie et al. (2003), for any time $t \geq 0$, we introduce an undirected graph $G(t)$ with vertex set $\{1, 2, \ldots, n\}$ where the vertex $i$ corresponds to the agent $i$ of the dynamical system described in the previous section. Furthermore, the vertices $i$ and $j$ of the graph where $i \neq j$ are connected by an edge if and only if the agents $i$ and $j$ are neighbours at time $t$: condition (2.1) holds. In order to formulate the main result of this paper, we introduce another undirected graph $\mathcal{G}$ with the same vertex set $\{1, 2, \ldots, n\}$, and the vertices $i$ and $j$ of the graph where $i \neq j$ are connected by an edge if and only if there exists a time sequence $t_s$ such that $t_s \to \infty$ and the agents $i$ and $j$ are neighbours at any time $t_0$:

$$
(x_j(t_0\delta) - x_i(t_0\delta))^2 + (y_j(t_0\delta) - y_i(t_0\delta))^2 \leq r^2
$$

for all $s = 1, 2, \ldots$. Any finite graph is a union of several non-intersecting nonempty connected subgraphs. Let $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m$ be such connected subgraphs of $\mathcal{G}$. Here $m \leq n$ and $m = 1$ in the case if the graph $\mathcal{G}$ is connected.

Furthermore, we will need the following assumption.

**Assumption 3.1**: The heading upper bound $a$ satisfies $a < \pi$.

Assumption 3.1 means that all agents velocity vectors always belong to some half-plane.

Now we are in a position to present the main results of this paper.

**Theorem 3.1**: Suppose that Assumption 3.1 holds. Then for any $h = 1, 2, \ldots, m$ there exists a heading $\hat{\theta}_h \in \Theta$ and an integer $T_h > 0$ such that

$$
\theta_i(t_0\delta) = \hat{\theta}_h \quad \forall t \geq T_h
$$

for all $i \in \mathcal{G}_h$.

Moreover, there exists an integer $T > 0$ such that

$$
G(t) = \mathcal{G} \quad \forall t \geq T.
$$

**Remark 3.1**: Theorem 3.1 shows that the heading of any agent is constant if time $t_0\delta$ is large enough. In other words, after a finite time, the set of $n$ agents always breaks into several groups that are separated from each other, and all agents from the same group have the same heading.

**Corollary 3.1**: If the graph $\mathcal{G}$ is connected, then there exists a heading $\hat{\theta} \in \Theta$ and an integer $T > 0$ such that

$$
\theta_i(t_0\delta) = \hat{\theta} \quad \forall t \geq T
$$

for all $i = 1, 2, \ldots, n$.

**Proof of Theorem 3.1**: Let $h \in \{1, 2, \ldots, m\}$ and consider all the agents corresponding to the subgraph $\mathcal{G}_h$. Let $\Theta_h^c \subset \Theta$ be the set of all $\theta \in \Theta$ such that $\theta_i(t_0\delta) = \theta$ for some $i \in \mathcal{G}_h$ and some time sequence $t_s \to \infty$. Furthermore, let

$$
\tilde{\theta} := \max_{\theta \in \Theta_h^c} \theta.
$$

(3.8)
Since \( \tilde{\theta} \in \Theta^\infty_h \), there exists at least one agent \( i \in G_h \) such that

\[
\theta_i(t_s\delta) = \tilde{\theta} \quad \forall s = 1, 2, \ldots
\]  

(3.9)

for some time sequence \( t_s \to \infty \).

In order to prove this theorem, we establish the following claim.

**Claim:** Condition (3.9) implies that there exists an integer \( T > 0 \) such that

\[
\theta_i(t\delta) = \tilde{\theta} \quad \forall t \geq T.
\]  

(3.10)

We prove this claim by contradiction. Indeed, if (3.10) does not hold, since the set \( \Theta \) is finite, there exists a sequence \( t_s(1) \to \infty \) such that

\[
\theta_i(t_s(1)\delta) = \tilde{\theta}^* \neq \tilde{\theta} \quad \forall s = 1, 2, \ldots
\]  

(3.11)

for some \( \tilde{\theta}^* \). It follows from (3.11) that \( \tilde{\theta}^* \in \Theta^\infty_h \). Now it follows from the definition of \( \tilde{\theta} \) (3.8) that \( \tilde{\theta}^* < \tilde{\theta} \). Furthermore, this, (3.11) and (3.9) imply that there exists a time sequence \( t_s(2) \to \infty \) such that

\[
\theta_i((t_s(2) + 1)\delta) = \tilde{\theta}, \quad \theta_i(t_s(2)\delta) \leq \tilde{\theta} - \frac{1}{M}  
\]  

(3.12)

for all \( s = 1, 2, \ldots \) From this and heading updating rules (2.2), (2.3), we obtain that for any time \( t_s(2) \) there exists an agent \( j(t_s(2)) \) with heading \( \theta_{j(t_s(2))}(t_s(2)\delta) \geq \tilde{\theta} + \frac{1}{M} \). Because the set of the agents and the set of possible headings are finite, there exists an index \( j \in G_h \), a heading \( \theta_0 > \tilde{\theta} \) and a subsequence \( t_s(3) \) of the sequence \( t_s(2) \to \infty \) such that

\[
\theta_j(t_s(3)\delta) = \theta_0
\]  

for all \( s = 1, 2, \ldots \) This implies that \( \theta_0 \in \Theta^\infty_h \). However, because \( \theta_0 > \tilde{\theta} \), this contradicts to (3.8). This completes the proof of the claim.

By our definition of \( \tilde{\theta} \), there exists at least one agent from \( G_h \) such that condition (3.9) holds. Hence, it follows from Claim that there exists at least one agent from \( G_h \) such that condition (3.10) holds. We now prove by contradiction that any agent \( j \in G_h \) satisfies (3.10). Indeed, assume that there exists an agent \( j \in G_h \) for which condition (3.10) does not hold. Then by Claim, this agent does not satisfy (3.9). Since the set \( \Theta \) is finite, it follows from the definition of the set \( \Theta^\infty_h \) that there exists an integer \( T > 0 \) such that

\[
\theta_j(t\delta) \in \Theta^\infty_h \quad \forall t \geq T.
\]  

(3.13)

Because condition (3.9) is not satisfied for this agent \( j \) and \( \tilde{\theta} \) is the largest element of the set \( \Theta^\infty_h \), there exists an integer \( T_\ast > 0 \) such that

\[
\theta_j(t\delta) \leq \tilde{\theta} - \frac{1}{M} \quad \forall t \geq T_\ast.
\]  

(3.14)
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Hence, we have proved that there exists an agent \( i \in G_h \) satisfying (3.10) and an agent \( j \in G_h \) satisfying (3.14). It is now obvious from (3.10), (3.14) and the dynamics equation (2.4) that

\[
(x_j(t\delta) - x_i(t\delta))^2 + (y_j(t\delta) - y_i(t\delta))^2 \to \infty
\]  

as \( t \to \infty \) for these two agents \( i \) and \( j \) from \( G_h \). Hence, condition (2.1) never holds for the agents \( i \) and \( j \). This means that the vertices \( i \) and \( j \) are not connected by an edge in the graph \( G \). In other words, we have proved that any agent \( i \in G_h \) satisfying (3.10) is not connected by an edge with any agent \( j \in G_h \) that does not satisfies (3.10). This implies that the subgraph \( G_h \) is not connected. Hence, we have proved that condition (3.10) holds for all agents from \( G_h \). Therefore, condition (3.6) holds. Finally, it is obvious that if (3.6) holds and the headings of all the agents are constant after some time, then the relationships between neighbours are also eventually constant, and condition (3.7) holds. This completes the proof of the theorem.

Now together with the graph \( G(0) \), introduce a undirected graph \( G^O(0) \) the vertices \( i \) and \( j \) of the graph \( i \neq j \) are connected by an edge if and only if the agents \( i \) and \( j \) are belong to an open disk with radius \( r \) at time 0:

\[
(x_j(0) - x_i(0))^2 + (y_j(0) - y_i(0))^2 < r^2.
\]  

(3.16)

The only difference between \( G(0) \) and \( G^O(0) \) is that the condition (3.16) is slightly stronger than (2.1).

We will need the following assumption.

**Assumption 3.2:** The graph \( G(0) \) is connected and \( G(0) = G^O(0) \).

**Theorem 3.2:** Consider the system (2.2), (2.3), (2.4) and suppose that Assumptions 3.1 and 3.2 hold. Then there exists a constant \( \epsilon > 0 \) such that for any \( \delta \leq \epsilon \) there exists a heading \( \hat{\theta} \in \Theta \) and an integer \( T > 0 \) such that

\[
\theta_i(t\delta) = \hat{\theta} \quad \forall t \geq T
\]  

(3.17)

for all \( i = 1, 2, \ldots, n \).

In order to prove Theorem 3.2, we need to establish a preliminary lemma.

**Lemma 3.1:** Let \( G \) be a connected undirected graph with \( n \) vertices, \( \theta_i(0) \in \Theta \) be some initial headings, \( i = 1, 2, \ldots, n \). Also, let \( N_i \) be the set of all vertices \( j \neq i \) connected to the vertex \( i \). For any \( t = 0, 1, \ldots \) define \( \theta_i(t + 1) \) by (2.2), (2.3) with \( \delta = 1 \) and \( N_i(t) = N_i \). Suppose that Assumption 3.1 holds. Then there exists a heading \( \hat{\theta} \in \Theta \) and a time \( T > 0 \) such that

\[
\theta_i(t) = \hat{\theta} \quad \forall t \geq T
\]  

(3.18)

for all \( i = 1, 2, \ldots, n \).
Proof of Lemma 3.1: Let $\Theta^\infty \subset \Theta$ be the set of all $\theta \in \Theta$ such that $\theta_i(t_s) = \theta$ for some $i$ and some time sequence $t_s \to \infty$. Furthermore, let

$$\tilde{\theta} := \max_{\theta \in \Theta^\infty} \theta.$$  \hfill (3.19)

Since $\tilde{\theta} \in \Theta^\infty$, there exists at least one vertex $i$ such that

$$\theta_i(t_s) = \tilde{\theta} \ \forall s = 1, 2, \ldots$$ \hfill (3.20)

for some time sequence $t_s \to \infty$. As in Claim from the proof of Theorem 3.1, we obtain that the condition (3.10) implies that there exists an integer $T > 0$ such that

$$\theta_i(t) = \tilde{\theta} \ \forall t \geq T.$$ \hfill (3.21)

Because $\tilde{\theta}$ is the largest element of the set $\Theta^\infty$, (3.21) implies that the set of all vertices of the graph $G$ can be divided into two non-intersecting subsets $V^+$ and $V^-$ where $V^+$ consists of vertices $i$ satisfying (3.21) and $V^-$ consists of vertices $j$ for which there exists an integer $T_* > 0$ such that

$$\theta_j(t) \leq \tilde{\theta} - \frac{1}{M} \ \forall t \geq T_*.$$ \hfill (3.22)

Now we prove that the set $V^-$ is empty. Indeed, because $G$ is connected, if $V^-$ is not empty there exist $i \in V^+$ and $j \in V^-$ connected by an edge. Therefore, by our construction, for all times $t \geq \max\{T, T_*\}$ all vertices $s$ from $N_i$ satisfies $\theta_i(t) \leq \tilde{\theta}$ and there is at least one vertex $j$ such that $\theta_j(t) \leq \tilde{\theta} - \frac{1}{M}$. However, this and the rules (2.5), (2.3) imply that $\theta_j(t) \leq \tilde{\theta} - \frac{1}{M}$ for all large enough $t$. This contradicts to (3.21). Hence the set $V^-$ is empty, therefore, (3.21) holds for all $i$. This completes the proof of the lemma.

Proof of Theorem 3.2: For any $\tau > 0$ introduce

$$\dot{x}_i(\tau) := x_i(0) + v\tau \cos(\theta_i(0)),$$

$$\dot{y}_i(\tau) := y_i(0) + v\tau \sin(\theta_i(0)).$$

Assumption 3.2 obviously implies that there exists a $\tau_0 > 0$ such that for $\tau \leq \tau_0$

condition

$$(\dot{x}_j(\tau) - \dot{x}_i(\tau))^2 + (\dot{y}_j(\tau) - \dot{y}_i(\tau))^2 \leq \epsilon^2$$

holds if and only if

$$(x_j(0) - x_i(0))^2 + (y_j(0) - y_i(0))^2 \leq \epsilon^2.$$ 

Furthermore, we apply Lemma 3.1 with $G = G(0)$ and take $T_*$ such that (3.18) holds. It is obvious that for any $\delta \leq \epsilon / T_*$ we obtain

$$\theta_i(T_*, \delta) = \tilde{\theta}$$

for some $\tilde{\theta}$ and all $i$. This completes the proof of Theorem 3.2.
4 Example of cyclic behaviour

In this section, we show that Assumption 3.1 is really crucial for our main result. Without this assumption the multi-particle system can oscillate in a bounded region of the plane over infinite time. This can be illustrated by the following example.

Let $n$ be an even integer, $n = 2k$. Consider a regular polygon with the vertices $p_1, p_2, \ldots, p_n$ defined by

$$p_i := \begin{pmatrix} \cos \left( \frac{2\pi i}{n} \right) \\ \sin \left( \frac{2\pi i}{n} \right) \end{pmatrix} \quad \forall i = 1, 2, \ldots, n.$$ 

From simple geometry, the distance between two consecutive vertices can be expressed as

$$|p_2 - p_1| = |p_3 - p_2| = \cdots = |p_n - p_{n-1}| = |p_1 - p_n| = 2 \sin \left( \frac{\pi}{n} \right).$$

Furthermore,

$$|p_3 - p_1| = |p_4 - p_2| = \cdots = |p_1 - p_{n-1}| = |p_2 - p_n| = 2 \sin \left( \frac{2\pi}{n} \right).$$

Let the radius $r$ be any number from the interval

$$2 \sin \left( \frac{\pi}{n} \right) < r < 2 \sin \left( \frac{2\pi}{n} \right). \quad (4.23)$$

Then we can take a small $v > 0$ such that

$$2 \sin \left( \frac{\pi}{n} \right) + 2v < r < 2 \sin \left( \frac{2\pi}{n} \right) - 2v. \quad (4.24)$$

Now suppose that at the initial time $t = 0$ our particles are located at the vertices of this regular polygon:

$$\begin{pmatrix} x_i(0) \\ y_i(0) \end{pmatrix} := p_i \quad \forall i = 1, 2, \ldots, n. \quad (4.25)$$

Furthermore, let $\theta_i(0) := 0$ if $i$ is odd, and $\theta_i(0) := \pi$ if $i$ is even. It follows from (4.23) that the disk of radius $r$ centred at a vertex contains two nearest neighbouring vertices and does not contain any other vertex of the polygon. Therefore, (2.2), (2.3), (2.4) imply that

$$\theta_i(\delta) := \pi, \quad \begin{pmatrix} x_i(\delta) \\ y_i(\delta) \end{pmatrix} := p_i + \begin{pmatrix} v \\ 0 \end{pmatrix}; \quad (4.26)$$

if $i$ is odd, and

$$\theta_i(\delta) := 0, \quad \begin{pmatrix} x_i(\delta) \\ y_i(\delta) \end{pmatrix} := p_i - \begin{pmatrix} v \\ 0 \end{pmatrix}; \quad (4.27)$$
if \( i \) is even. Furthermore, from this (2.2), (2.3), (2.4) and (4.24) we obtain

\[
\begin{pmatrix}
 x_i(2\delta) \\
y_i(2\delta)
\end{pmatrix} := p_i \quad \forall i = 1, 2, \ldots, n,
\]

and \( \theta_i(0) := 0 \) if \( i \) is odd, and \( \theta_i(0) := \pi \) if \( i \) is even. Now from (4.25), (4.26), (4.27) and (4.28) we obtain that the system is periodic with the period \( 2\delta \). The graph \( \mathcal{G} \) is obviously connected, but the agents will never go in any common direction.

5 Conclusions

We describe conditions under which a group of autonomous mobile robots will move in the same directions. We give an example of cyclic dynamics in the situation where one of our assumptions does not hold.

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References


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