ON CRUM'S PROBLEM

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In this article I give a solution of the following problem of M. Crum.

What is the maximum number of convex polyhedra, non-overlapping and such that any pair of them have a common boundary of positive area?

The answer to the similar plane problem is "four" and it was expected that a finite, rather small number, ten or twelve, would be the answer to the above problem. I shall show that actually the answer is "infinity".

Take rectangular coordinate axes and in the vertical plane $XOZ$ draw a polygonal line $A_0A_1A_2...$, from $A_0(1, 0, 0)$, above $OX$, convex downwards, of total length less than $\frac{1}{4}$ and such that the angle between the directions $OX$ and $A_nA_{n+1}$ is greater than $\frac{s}{n}$ for any $n$.

Now take a sequence of positive numbers $\{\delta_n\}$ such that $\sum \delta_n < \frac{1}{4}$.

Join the point $B_0(0, 1, 0)$ to $A_0$ and take the point $D_1$ on $B_0A_0$ such that $B_0D_1 = \delta_1$; join $D_1$ to $A_1$ and take the point $D_2$ on $D_1A_1$ such that $D_1D_2 = \delta_2$; then join $D_2$ to $A_2$ and take the point $D_3$ on $D_2A_2$ such that $D_2D_3 = \delta_3$, and so on. Denote by $r_0$ the plane $XOY$ and by $r_n$, for $n = 1, 2, 3, ...$, the plane through $A_{n-1}, D_n, A_n$. Let $B_n$ and $C_n$ be respectively the points of intersection of $r_n$ with $OY$ and $OZ$. It is easy to see that every $B_n$ is on the positive half of the $Y$-axis.

Denote by $S_{k+1}$, $k = 0, 1, 2, ...$, the polyhedron consisting of the points of the first octant that are not below any one of the planes $r_0, r_1, ..., r_k$ and not above the plane $r_{k+1}$.

If we denote by $x^+, y^+$ the half-spaces $x \geq 0, y \geq 0$, by $r_n^+$ the half-space of points on and above $r_n$, and by $r_n^-$ the half-space of points on or below $r_n$, then $S_{k+1} = x^+y^+r_0^+r_1^+...r_k^+r_{k+1}^-$. Being the intersection of half-spaces, $S_{k+1}$ is a convex polyhedron. By a direct inspection we see that the triangle $D_{k+1}A_{k+1}C_{k+1}$ belongs to the common boundary of $S_{k+1}$ and $S_{k+2}$, and thus $S_{k+1}$ and $S_{k+2}$ satisfy the required condition. Denote the points of intersection of $r_k$, $k > 2$, with the lines $B_0A_0, D_1A_1, D_2A_2, ...$ by $E_{0k}, E_{1k}, E_{2k}, ...$ respectively.

The triangle $A_1D_1A_0$ obviously forms a part of the surface of $S_1$. We also have

\[ \Delta A_1D_1A_0 \subset r_0^+r_1^+r_2^+. \]

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The planes $r_3, r_4, ..., r_k, ...$ meet the triangle $A_1D_1A_0$ in the lines $D_2E_{03}, E_{14}E_{04}, ..., E_{1k}E_{ok}, ...$ respectively, and the part of $A_1D_1A_0$ to the left of $E_{1k}E_{ok}$ is in $r_k^-$, and the one to the right in $r_k^+$; whence

$$D_2D_1E_{03} \subset r_3^-, \quad E_{14}D_2E_{03}E_{04} \subset r_3^+ r_4^-, \quad E_{15}E_{14}E_{04}E_{05} \subset r_3^+ r_4^+ r_5^-, \ ...,$$

and, by (1),

$$D_2D_1E_{03} \subset S_3, \quad E_{14}D_2E_{03}E_{04} \subset S_4, \quad E_{15}E_{14}E_{04}E_{05} \subset S_5, \ ... .$$

Thus $S_1$ has a common boundary of positive measure with any other $S_k$.

By considering the triangles $A_2D_2A_1, A_3D_3A_2, ...$ we shall come to a similar conclusion with respect to $S_2, S_3, ...$.

Thus any pair of polyhedra of the infinite sequence $\{S_k\}$ satisfy the required conditions.

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A SEQUENCE OF POLYHEDRA HAVING INTERSECTIONS OF SPECIFIED DIMENSIONS

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1. M. Crum proposed the following problem. What is the maximum number of non-overlapping convex polyhedra in 3-space which have the property that any two have a two-dimensional intersection? Besicovitch† proved that the answer is infinity, by constructing an infinite sequence of non-overlapping convex polyhedra any two of which have a two-dimensional intersection. In the present note I generalize this result by constructing a sequence of polyhedra $S_1, S_2, ...$ in $n$-space which have the property that, if $1 \leq k \leq \frac{1}{2}(n+1)$, any $k$ of the $S_m$ have an $(n-k+1)$-dimensional intersection. It is known that for $n = 1$ and for $n = 2$ the number $\frac{1}{2}(n+1)$ cannot be replaced by $\frac{1}{2}(n+1)+1$. All $S_{\mu}$ of our construction will lie in a fixed cube of side $2^{n+2}$, and $S_m$, being the intersection of $2n+m+1$ half-spaces, will be a convex polyhedron.

THEOREM. Let $n$ be a positive integer. Let

$$(1) \quad 0 < t_0 < t_1 < ...; \quad t_\mu < 1 \quad (\mu > 0).$$

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