Spectral Element Approach for Coupled Radiative and Conductive Heat Transfer in Semitransparent Medium

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1 Introduction
The coupled problem of radiative and conductive heat transfer is of considerable practical importance in engineering applications. These applications include, for example, the analysis of heat transfer process in glass and industrial furnaces or the analysis of thermal performance of porous insulating materials, such as fibers, powders, foams, and so on. The solution of this coupled problem involves the evaluation of radiative heat source and the solution of heat diffusion equation. Due to the inherent complexity associated with radiative transfer, the evaluation of radiative heat source is the main difficulty for the solution of coupled radiative and conductive heat transfer problems in semitransparent medium.

Recently, many numerical methods have been proposed to solve the coupled radiative and conductive heat transfer in semitransparent medium. Viskanta and Grosh [1] analyzed combined radiation and conduction heat transfer between one-dimensional parallel plates filled with absorbing medium using iterative method. Yuen and Wong [2] used a successive approximation technique to solve the combined conductive and radiative heat transfer in one-dimensional absorbing, emitting, and anisotropically scattering medium. Burns et al. [3] analyzed numerically the problem of coupled radiative and conductive heat transfer in one-dimensional absorbing medium using traditional Galerkin finite element method and the Swartz–Wendroff approximation. Yuen and Takara [4] analyzed coupled radiative and conductive heat transfer in two dimensional rectangular enclosure by using a class of generalized exponential integral function. Razzaque et al. [5] employed the finite element method to solve the coupled radiation and conduction in a two dimensional rectangular enclosure filled with gray medium. Kim and Baek [6] studied the coupled radiative and conductive heat transfer in two dimensional rectangular enclosure, in which the central difference scheme was used to discretize the heat diffusion equation and the discrete ordinate method was employed to solve the radiative transfer equation. Sakami et al. [7] solved the coupled radiative and conductive heat transfer in two dimensional complex geometries filled with absorbing, emitting, and scattering medium, in which a modified discrete ordinate approach was applied to solve the radiative transfer equation, while the heat diffusion equation was solved by finite element method. Talukdar and Mishra [8] solved the combined conduction and radiation problem in one-dimensional gray absorbing, emitting, and anisotropically scattering medium using the collapsed dimension method (CDM). Mishra and Lankadasu [9] solved the transient conductive and radiative heat transfer in two dimensional rectangular enclosure filled with absorbing, emitting, and scattering medium, in which the energy equation of the problem is solved with the lattice Boltzmann method (LBM), while the radiative transfer equation is solved using the CDM. Besides that different discretization schemes are often used to solve the heat diffusion equation and the radiative transfer equation, most of these methods are lower order method (first order or second order) and just offer h convergence, i.e., the convergence gained by reducing the element size h or h refinement. As a result, remeshing or refining is often needed in order to gain the wanted accuracy.

The well known radiative transfer equation (RTE), which describes the transport of radiation intensity through an absorbing, emitting, and scattering medium, is one of the first order integro-differential equations and can be written in Cartesian coordinates as

\[
\nabla \cdot \mathbf{I} + \beta I = \kappa I + \frac{\sigma_T}{4\pi} \int_{4\pi} I(x,\Omega') \Phi(\Omega',\Omega) d\Omega' \quad (1)
\]

where \(\Omega\) is the unit direction vector of radiation, \(\beta\) is the extinction coefficient, \(\kappa\) is the absorption coefficient, \(\sigma_T\) is the scattering coefficient, and \(\Phi\) is the scattering phase function. The first term
of the left hand side of Eq. (1) can be seen as a convection term with convection velocity of \( \Omega \). Because of the vanishing of diffusion term, the RTE can be considered as a convection dominated equation. This is the major difference between the RTE and the heat diffusion equation. The solution of the RTE often requires a different solver instead of the one used in the solution of the heat diffusion equation and thus makes it inconvenient in the coupled conduction and radiation analysis. The presence of convection term may cause nonphysical oscillatory of solutions. This type of instability occurs in many numerical methods including finite difference method and finite element method if no special stability treatment is taken. Special stabilization techniques, such as upwinding scheme or artificial viscosity, are often used in finite volume method (FVM) and finite element method (FEM). Besides taking various numerical stabilization schemes, another method to overcome the stability problem is to analytically transform the original RTE into a numerically more stable equation, for example, the second order partial differential equation. One famous transformed equation is the even parity formulation (EPF) of the RTE, which is a second order partial differential equation of the even parity of radiative intensity. It is well known that the second order derivative term have diffusive characteristic and good numerical properties. Cheong and Song [10] examined several spatial discretization schemes in the discrete ordinate solution of the EPF. Freeland and Jessee [11,12] studied the finite element solution of the EPF. Though the stability of the finite element solution of the second order even parity equation is ensured, numerical results indicate that the solution obtained using the FEM is less accurate as the optical thickness and the wall emissivity are increased.

Recently, Zhao and Liu [13] derived a second order primitive variable radiative transfer equation (SORTe), which is in a form similar to the heat diffusion equation. Numerical results show that the SORTE overcomes most of the drawbacks of EPF and can be effectively applied to solve radiative transfer in absorbing, emitting, and anisotropically scattering medium. The FEM based on the SORTE is numerically stable, efficient, and accurate, and presents a symmetric stiff matrix. It is expected that, based on the SORTE, a unified solver is allowed to stably solve the coupled radiative and conductive heat transfer process in multidimensional semitransparent medium.

Spectral element method, originally proposed by Patera [14] for the solution of fluid problem, combines the competitive advantages of high order spectral method and FEM. Spectral method is preferred because of its exponential convergence characteristics. FEM is attractive because of its highly flexible domain decomposition capability, which is extremely useful for the problem with complex geometries. Spectral element method provides basically two types of refinement scheme to achieve convergence. The most common one is to use smaller size elements in the regions where higher resolution is required, which is called \( h \) refinement in literature because the element sizes are usually denoted by \( h \). The second type refinement is the \( p \) refinement, in which the number of elements and their sizes are kept the same but the approximation order inside the elements is increased where higher resolution is required. As a result, spectral element method is very flexible for solving the multidimensional problem in complex geometries with higher order accuracy. In recent years, spectral element method has attracted the interest of many researchers in the physics and engineering communities [15–22].

In order to best combine both the advantages of the SORTE and the spectral element approach, in this paper, a spectral element method (SEM) based on the SORTE is developed to solve the coupled radiative and conductive heat transfer in multidimensional semitransparent medium. With application of the advantages of the SORTE, the radiative transfer and the heat diffusion equations are both solved by spectral element approach. Four cases of coupled radiative and conductive heat transfer in semitransparent medium are taken to verify the performance of the presented method.

2 Mathematical Formulation

2.1 Second Order Radiative Transfer Equation. The SORTE was first presented and detailed analyzed in a recent paper [13]. Equation (1) can be written formally as

\[
I = \beta^{-1} \left[ -\Omega \cdot \nabla I + \kappa I + \sigma_T \frac{\alpha_I}{4\pi} \int_{\Omega'} I(r, \Omega') \Phi(\Omega, \Omega') d\Omega' \right] 
\]

Substituting Eq. (2) back into the derivative term of Eq. (1) leads to the second order partial differential equation as follows:

\[
-\Omega \cdot \nabla (\beta^{-1} \Omega \cdot \nabla I) + \beta I = \beta S - \Omega \cdot \nabla S
\]

For the opaque, diffusely emitting and reflecting walls, the boundary conditions are given for both the inflow and outflow boundaries as

\[
I(r_w, \Omega) = \varepsilon_w I_s(r_w) + \frac{1 - \varepsilon_w}{\pi} \int_{\Omega} I_0(r, \Omega') \left. \frac{1}{n_w \cdot \Omega} \right|_{\Omega = \Omega_w} \times |n_w \cdot \Omega| = |D| d\Omega' \quad n_w \cdot \Omega > 0 \quad (5a)
\]

\[
\beta^{-1} \Omega \cdot \nabla I(r_w, \Omega) + I(r_w, \Omega) = S(r_w, \Omega) \quad n_w \cdot \Omega < 0 \quad (5b)
\]

where \( \varepsilon_w \) is the wall emissivity, and \( n_w \) is the unit inward normal vector of boundary. A schematic of the prescribed boundary conditions for the SORTE is shown in Fig. 1 for a clear view, where \( \Gamma \) denotes all the boundary of solution domain with \( \Gamma = \Gamma_D \cup \Gamma_N \). \( \Gamma_D \) and \( \Gamma_N \) denote the inflow boundary (Dirichlet type) and the outflow boundary (Neumann type), respectively.

The discrete ordinate equation of the SORTE given by Eq. (3) can be written as

\[
-\Omega^m \cdot \nabla [\beta^{-1} \Omega^m \cdot \nabla I(r, \Omega^m)] + \beta I(r, \Omega^m) = \beta S(r, \Omega^m)
\]

\[
-\Omega^m \cdot \nabla S(r, \Omega^m)
\]

with the following inflow and outflow boundary conditions:

\[
I(r_w, \Omega^m) = \varepsilon_w I_s(r_w) + \frac{1 - \varepsilon_w}{\pi} \sum_{\Omega^m} I(r_w, \Omega^m) \bigg|_{n_w \cdot \Omega^m} \times |n_w \cdot \Omega^m| = |D| d\Omega' \quad n_w \cdot \Omega^m > 0 \quad (7a)
\]
where $\Omega^m$ is the discrete angular direction and $w^m$ is the corresponding angular quadrature weight.

2.2 Energy Equation for Coupled Radiative and Conductive Heat Transfer Processes. In the following analysis, Eq. (6) with boundary conditions, Eq. (7), is taken as the model equations to calculate radiative intensity field. When coupled analysis of radiative and conductive heat transfer is needed, an additional equation, namely, heat diffusion equation is required to be solved simultaneously. The energy equation for a steady state coupled conduction-radiation problem can be written as

$$\nabla \cdot [k \nabla T(r)] = \nabla \cdot q_i(r)$$

with boundary condition

$$T(r_w) = \bar{T}(r_w) \quad r_w \in \Gamma_T$$

$$-k n_w \cdot \nabla T(r_w) = \bar{q}(r_w) \quad r_w \in \Gamma_q$$

where $\bar{T}$ denotes the temperature distribution on Dirichlet boundary $\Gamma_T$, $\bar{q}$ denotes the conductive outflow heat flux distribution at Neumann boundary $\Gamma_q$. $q_i(r)$ is the radiative heat flux, and the $\nabla \cdot q_i(r)$ is the radiative energy source given by

$$\nabla \cdot q_i(r) = \kappa \left[ 4\pi I_b - \int_{4\pi} I(r, \Omega) d\Omega \right]$$

2.3 Galerkin Weak Formulation. The discrete ordinate equation of the SORTE (Eq. (6)) weighted by $W(r)$ and integrated over the solution domain after using Gauss divergence theorem leads to

$$\langle \beta^{-1} \Omega^m \cdot \nabla I(r_w, \Omega^m) + I(r_w, \Omega^m), n_w \cdot \Omega^m \rangle - \kappa n_w \cdot \nabla T(r_w)\rangle = \langle \bar{q}(r_w), n_w \rangle / \Omega^m, n_w \rangle_{\Gamma_q} + \langle \beta W, W \rangle$$

$$= \langle \beta S^m, W \rangle - \langle \Omega^m \cdot \nabla S^m, W \rangle$$

where $n_w$ is the inward normal vector of the boundary, $\Gamma$ denotes the boundary of solution domain ($\Gamma = \Gamma_D \cup \Gamma_N$), $\Gamma_D$ and $\Gamma_N$ denotes the inflow boundary, and outflow boundary, as shown in Fig. 1. The operators $\langle , \rangle$ and $\langle , \rangle$ are defined as follows:

$$\langle f, g \rangle = \int_{4\pi} fg dA \quad \langle f, g \rangle_{\Gamma} = \int_{\Gamma} fg dA$$

Assuming the weight function $W(r)$ is zero on the inflow boundary $\Gamma_D$, then Eq. (15) can be written as

$$\langle \beta^{-1} \Omega^m \cdot \nabla I(r_w, \Omega^m), W \rangle + \langle \beta^{-1} \Omega^m \cdot \nabla W, W \rangle_{\Gamma_q} + \langle \beta W, W \rangle$$

$$= \langle \beta S^m, W \rangle - \langle \Omega^m \cdot \nabla S^m, W \rangle$$

Considering the outflow boundary condition given by Eq. (7b), we get

$$\langle \beta^{-1} \Omega^m \cdot \nabla I(r_w, \Omega^m), n_w \rangle_{\Gamma_q} = \langle S^m - I^m, W \rangle_{\Omega^m, n_w} \rangle_{\Gamma_q}$$

Substituting Eq. (13) into Eq. (12) leads to the following weak formulation:

$$\langle \beta^{-1} \Omega^m \cdot \nabla I(r_w, \Omega^m), W \rangle - \langle I^m, W \rangle_{\Omega^m, n_w} \rangle_{\Gamma_q} + \langle \beta W, W \rangle$$

$$= \langle \beta S^m, W \rangle - \langle \Omega^m \cdot \nabla S^m, W \rangle - \langle \Omega^m \cdot \Omega^m, n_w \rangle_{\Gamma_q}$$

The heat diffusion equation (Eq. (8)) is a kind of classical elliptical equation, and its Galerkin weak form is given as

$$- \langle k \nabla T, \nabla W \rangle = \langle \nabla \cdot q_i(r), W \rangle + \langle \bar{q}, W \rangle_{\Gamma_q}$$

$$= \kappa \langle 4\pi I_b - \int_{4\pi} I(r, \Omega) d\Omega, W \rangle + \langle \bar{q}, W \rangle_{\Gamma_q}$$

2.4 Spectral Element Approach. The unknown radiative intensity and temperature can be approximated by nodal basis function with Kronecker delta property as

$$I^m(r) = \sum_{i=1}^{N_\text{sol}} I_i^m \phi_i(r)$$

$$T(r) = \sum_{i=1}^{N_\text{sol}} T_i \phi_i(r)$$

where $\phi_i(r)$ is the nodal basis function, $I_i^m$ denotes radiative intensity of direction $\Omega_i^m$ at solution nodes $i$, $T_i$ denotes temperature at solution nodes $i$, and $N_\text{sol}$ is the total number of solution nodes. In SEM, the nodal basis function $\phi_i(r)$ is constructed on each element by orthogonal polynomial expansion. In this paper, Chebyshev polynomial expansion is used. The details about building global nodal basis function were described in Ref. [24].

By substituting the approximation of radiative intensity (Eq. (17)) into the Galerkin weak formulation of SORTE (Eq. (15)) and taking the weight function $W(r)$ as nodal basis function $\phi_i(r)$, the spectral element discretization of SORTE can be written in matrix form as

$$K^m T^m = H^m$$

Here, the matrices $K^m$ and $H^m$ are defined as

$$K^m_{ij} = \langle \beta^{-1} \Omega^m \cdot \nabla \phi_i, \Omega^m \cdot \nabla \phi_j \rangle - \langle \phi_i, \phi_j \Omega^m \cdot n_w \rangle_{\Gamma_q} + \langle \beta \phi_i, \phi_j \rangle$$

$$H^m_{ij} = \langle \beta S^m, \phi_i \rangle - \langle \Omega^m \cdot \nabla S^m, \phi_j \rangle - \langle \beta^m, \Omega^m \cdot n_w \rangle_{\Gamma_q}$$

It can be seen from Eq. (20) that the stiffness matrix $K^m$ is symmetric for every discrete direction.

Similarly, the spectral element discretization of heat diffusion equation can be written in matrix form as

$$M T = N$$

Here, the matrices $M$ and $N$ are defined as

$$M_{ij} = -\langle k \nabla \phi_i, \nabla \phi_j \rangle$$

$$N_j = \kappa \langle 4\pi I_b - \int_{4\pi} I(r, \Omega) d\Omega, \phi_j \rangle + \langle \bar{q}, \phi_j \rangle_{\Gamma_q}$$

2.5 Solution Procedure. The implementation of the SEM for analysis of coupled radiative and conductive heat transfer is carried out according to the following procedure.

(a) Step 1. Mesh the solution domain with $N_\text{id}$ nonoverlap elements and generate global unique spectral nodes on each element according to the order $p$ of polynomial expansion.

(b) Step 2. Assume initial radiative intensity field and temperature field.

(c) Step 3. Assemble the matrices $M$ and $N$ for the discretized heat diffusion equation. Impose Dirichlet boundary condition and solve the discretized equation to update the temperature field. Make under-relaxation on the temperature field as needed.

(d) Step 4. Loop each angular direction for $m=1,...,M$ and assemble the matrices $K^m$ and $H^m$ for the discretized
considered as the benchmark solution. Note that the temperature values of Planck number are studied. Figure 2 shows the solution of dimensionless temperature distribution for different values of Planck number are less than 0.2%.

In general, the maximum deviation between these results is good. In general, the maximum deviation between these results is quite small. The agreement between the different results is quite good. The maximum deviation between these results is less than 0.2%.

In this paper, the maximum relative error $10^{-4}$ of temperature $\left(\|T_{\text{new}} - T_{\text{old}}\|/\|T_{\text{new}}\|\right)$ is taken as stopping criterion for the global iteration.

3 Results and Discussion

Computer code is developed based on the numerical method described above. To verify the formulations presented in this paper, three various test cases are selected to verify the performance of the new numerical method for coupled radiative and conductive heat transfer in semitransparent medium. For the sake of quantitative comparison with the benchmark solutions, the integral averaged relative error of SEM solution is defined as

$$\text{relative Error %} = \frac{\int |\text{SEM solution}(x) - \text{benchmark result}(x)|dx}{\int |\text{benchmark result}(x)|dx} \times 100 \quad (23)$$

3.1 Case 1: One-Dimensional Nonscattering Gray Medium Between Parallel Black Plates. We consider the coupled radiative and conductive heat transfer in a layer of absorbing-emitting medium between infinite parallel black plates. The medium between the plates is gray and has a constant thermal conductivity and absorption coefficient. The temperatures of the plates are $T_0$ at $x/L=0.0$, and $T_f=0.1T_0$ at $x/L=1.0$, respectively. The optical thickness of the layer is $\tau = \beta L = 1.0$. The dimensionless temperature distribution within the layer is determined for different values of Planck number $N_{plk}=0, 0.01, 0.1, 1.0$. This case was considered as a benchmark example and studied by several researchers. The dimensionless temperature distributions obtained using the SEM are shown in Table 1 and compared to the results of Burns et al. [3] and Nice [25] for $N_{plk}=0.01, 0.1, 1.0$. Here, $S_8$ approximation is used for angular discretization and two elements with tenth order polynomial expansion are used for spatial discretization on each element. The agreement between the different results is quite good. In general, the maximum deviation between these results is less than 0.2%.

The $h$- and the $p$-convergence characteristics of the SEM for the solution of dimensionless temperature distribution for different values of Planck number are studied. Figure 2 shows the $p$-convergence characteristics of the SEM, where two elements are used and the solution obtained with 20th order polynomial is considered as the benchmark solution. Note that the temperature distribution for $N_{plk}=0$ is solved as a purely radiation equilibrium problem and without coupling with the heat diffusion equation. It can be seen that the convergence is very fast for different values of Planck number and the convergence rate follows the exponential law. With the increase of Planck number, the convergence rate tends to increase. Figure 3 shows the $h$-convergence characteristics of the SEM. Here, the first order polynomial is used for all the values of Planck number and the same benchmark solution is used for comparison. Similarly, the solution accuracy is better for a relative greater Planck number under the same spatial discretization for the coupled solution. A detailed comparison of the $h$- and the $p$-convergence rate is shown in Fig. 4, where the horizontal coordinate denotes the total number of solution nodes for spatial discretization.

Table 1 Comparison of dimensionless temperature distributions obtained from different methods

<table>
<thead>
<tr>
<th>$x/L$</th>
<th>$N_{plk}=0.01$</th>
<th>$N_{plk}=1.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.9230</td>
<td>0.9206</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8919</td>
<td>0.8888</td>
</tr>
<tr>
<td>0.3</td>
<td>0.8680</td>
<td>0.8650</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8443</td>
<td>0.8415</td>
</tr>
<tr>
<td>0.5</td>
<td>0.8182</td>
<td>0.8156</td>
</tr>
<tr>
<td>0.6</td>
<td>0.7859</td>
<td>0.7837</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7379</td>
<td>0.7361</td>
</tr>
<tr>
<td>0.8</td>
<td>0.6477</td>
<td>0.6460</td>
</tr>
<tr>
<td>0.9</td>
<td>0.4570</td>
<td>0.4557</td>
</tr>
</tbody>
</table>

Fig. 2 The $p$-convergence characteristics of the SEM for different values of Planck number

Fig. 3 The $h$-convergence characteristics of the SEM for different values of Planck number
discretization and only results for $N_{plk}=0.01$ and 0 is plotted for clarity. For the one-dimensional problem considered here, total number of solution nodes $N_{sol}$ can be written as a function of the total number of element $N_3$ and the order of polynomial expansion $p$ as $N_{sol}=pN_3+1$. In the $h$-refinement analysis, the first order polynomial expansion is used and only the number of elements is increased during refining. In the $p$-refinement analysis, two elements are used and only the order of polynomial is increased during refining. By comparison, it is clearly shown that the convergence rate of $p$ refinement is superior to that of $h$ refinement. This implies that more accuracy of solution can be obtained by $p$ refinement than $h$ refinement under the same computational cost.

3.2 Case 2: One-Dimensional Anisotropically Scattering Gray Medium Between Parallel Black Plates. We consider the coupled radiative and conductive heat transfer in a layer of anisotropically medium between the infinite parallel black plates. The temperatures of the plates are $T_0$ at $x/L=0.0$, and $T_L=\alpha T_0$ at $x/L=1.0$, respectively. The optical thickness $\tau_2$ of the layer is 1.0. The medium has a linear anisotropically scattering phase function: $\Phi(\mu,\mu')=1+\alpha_1(\mu\mu')$. This case was also studied by Tulukdar and Mishra [8] using the CDM. The SEM is applied to this case to study its performance for solution of coupled radiative and conductive heat transfer in scattering media. Here, the $S_{8}$ approximation is used for angular discretization and two elements with tenth order polynomial expansion are used for spatial discretization on each element. Figure 5 shows the dimensionless temperature distributions obtained by the SEM for different scattering conditions, namely, isotropically scattering ($\alpha_1=0$) and linear anisotropically scattering ($\alpha_1=1$), and different values of Planck number. By comparison, the results of the SEM agree very well with those obtained by the CDM [8]. Figure 6 shows the $h$- and the $p$-convergence curves of the SEM for the solution of dimensionless temperature distribution with $N_{plk}=0.01$, $\alpha_1=0.1$, $\omega=0.5$, and different values of $a_1$. The solution obtained with two elements and 20th order polynomial is considered as the benchmark solution. For study of the $p$-convergence characteristics, two elements are used for all $p$-refinement schemes. For study of the $h$-convergence characteristics, the first order polynomial is used for all the $h$-refinement schemes. It can be seen that the convergence rate of $p$ refinement follows exponential law and is superior to that of $h$ refinement when the medium is anisotropically scattering or not.

3.3 Case 3: Two Dimensional Absorbing-Emitting Medium in a Black Enclosure. In this case, we consider the coupled radiative and conductive heat transfer in a two dimensional square gray semitransparent medium enclosed by black walls. The medium is nonscattering and the optical thickness based on the side length $L$ of square enclosure is $\tau_L=BL=1.0$. The temperature of the left wall is maintained at $T_{w1}=1000$ K, while the other walls are kept at 500 K. The SEM is applied to solve the dimensionless temperature distribution within the enclosure. The $S_{8}$ approximation is used for the angular discretization. The enclosure is discretized uniformly into four quadrilateral elements and sixth order polynomial expansion is used on each element. The dimensionless temperature distribution along the symmetry line ($y/L=0.5$) are presented in Fig. 7 for three values of Planck number $N_{plk}$.
The tolerance for skewed grid of the SEM is tested using this case. The set of skewed mesh with increasing degree of skewness is shown in Fig. 8 from (a) to (d). As shown in Fig. 8, the degree of skewness $s_1$ is defined as the ratio of the length of diagonal line $b = h/a$ of the center diamond element. By definition, the skewness of the set of meshes from (a) to (b) is 1.0, 2.6, 5.6, and 11.3, respectively. Here, dimensionless temperature distribution along the symmetry line obtained using the mesh (a) of Fig. 8, namely, $s_1=1.0$, which is the least skewed one, with sixth order polynomial expansion on each element is taken as the benchmark solution to verify the results obtained using other meshes. Figure 9 shows the relative deviation between the benchmark solution and those obtained using the skewed meshes. As shown in Fig. 9, the relative deviation increases with the degree of skewness but decreases quickly with the increase of the order of polynomial. This implies that SEM has good property to tolerate the skewed meshes. The ability of tolerance for skewed meshes can be effectively promoted by $p$ refinement. In general, the maximum deviation between these results is less than 0.6%.

3.4 Case 4: Coupled Radiative and Conductive Heat Transfer in a Gray Circular Ring. In this case, the coupled radiative and conductive heat transfer in a circular ring is studied. The configuration of the circular ring is shown in Fig. 10. The radius inner circle ($R_1$) is half of the radius of the outer circle ($R_2$). The circular ring is filled with isotropically scattering medium. The wall emissivities of the enclosure are 0.5. The temperature of the inner circle ($T_1$) and the outer circle ($T_2$) are kept as 1000 and 100 K, respectively. The optically thickness based on radius of the outer circle is $\tau_1 = \beta R_1 = 2$, and the Planck number is $N_{Plk} = k\beta / (4\sigma T_1^3) = 0.03$. The circular ring is discretized into 60 elements, and $S_4$ approximation is used for angular discretization. The dimensionless temperature distribution along the radius is solved by the SEM with two different spatial decomposition schemes, namely, a good quality mesh shown in Fig. 10(a) and a poor quality mesh shown in Fig. 10(b). Figure 11 shows the dimensionless temperature distribution obtained by the SEM for three different scattering albedos, namely, $\omega = 0$, 0.5, and 1.0 and compared to the results obtained by Sakami et al. [7] using a
The convergence rate of the Planck number and follows the exponential law. By comparison, the quality of mesh.

coupled radiative and conductive heat transfer in semitransparent medium, which best combines both the advantages of the SORTE and the spectral element approach. The RTE and the heat diffusion equation are both solved by spectral element approach. The resulting stiffness matrices of both the discretized radiative transfer and heat diffusion equations are symmetric. Four various test problems were taken as examples to verify the performance of the presented method. The predicted dimensionless temperature distributions agree well with the benchmark solutions in references. The $h$-convergence and the $p$-convergence characteristics of the SEM are studied. The convergence rate of $p$ refinement is very fast for different values of Planck number and follows the exponential law. By comparison, the convergence rate of $p$ refinement is superior to that of $h$ refinement. The SEM has good property to tolerate skewed meshes and its ability to tolerate skewed meshes can be effectively promoted by $p$ refinement. The SEM is very effective to solve coupled radiative and conductive heat transfer in semitransparent medium with complex configurations and demands little on the quality of mesh.

Acknowledgment

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4 Conclusions

The highlight of the SORTE is that, it overcomes the nonphysical oscillation of the solutions of RIE when solved by FEM and allows the radiative transfer and the heat diffusion equations to be solved by a unified solver. In this paper, based on the SORTE, a SEM is employed to solve coupled radiative and conductive heat transfer in semitransparent medium, which best combines both the advantages of the SORTE and the spectral element approach. The resulting stiffness matrices of both the discretized radiative transfer and heat diffusion equations are symmetric. Four various test problems were taken as examples to verify the performance of the presented method. The predicted dimensionless temperature distributions agree well with the benchmark solutions in references. The $h$-convergence and the $p$-convergence characteristics of the SEM are studied. The convergence rate of $p$ refinement is very fast for different values of Planck number and follows the exponential law. By comparison, the convergence rate of $p$ refinement is superior to that of $h$ refinement. The SEM has good property to tolerate skewed meshes and its ability to tolerate skewed meshes can be effectively promoted by $p$ refinement. The SEM is very effective to solve coupled radiative and conductive heat transfer in semitransparent medium with complex configurations and demands little on the quality of mesh.

Nomenclature

\begin{align*}
\alpha_l & = \text{anisotropy factor} \\
h & = \text{element size} \\
H & = \text{matrix defined in Eq. (20b)} \\
I & = \text{radiative intensity, W/} (\text{m}^2) \\
I_b & = \text{Blackbody radiative intensity, W/} (\text{m}^2) \\
k & = \text{Thermal conductivity coefficient, W/} (\text{Km}) \\
K & = \text{stiff matrix defined in Eq. (20a)} \\
L & = \text{side length of square enclosure} \\
M & = \text{number of discrete ordinate direction} \\
M & = \text{stiff matrix defined in Eq. (22a)} \\
N & = \text{unit inward normal vector} \\
N_{el} & = \text{total number of elements} \\
N_{Plk} & = \text{Planck number, } N_{Plk} = k\beta/(4\alpha T^4) \\
N_{sol} & = \text{total number of solution nodes} \\
p & = \text{order of polynomial expansion} \\
q_r & = \text{radiative heat flux vector} \\
q & = \text{conductive outflow heat flux at Neumann boundary } \Gamma_r, \text{ W/m}^2 \\
r & = \text{Spatial coordinate vector} \\
s & = \text{degree of element skewness} \\
S & = \text{source function defined in Eq. (4)} \\
T & = \text{temperature, K} \\
\bar{T} & = \text{temperature at the Dirichlet boundary } \Gamma_r, \text{ K} \\
V & = \text{computational domain} \\
w & = \text{weight of discrete ordinate approximation} \\
W & = \text{weight function} \\
x, y, z & = \text{Cartesian coordinates} \\
\beta & = \text{extinction coefficient } \beta = (\kappa + \sigma_s), \text{ m}^{-1} \\
\epsilon_w & = \text{wall emissivity} \\
\phi & = \text{nodal basis function} \\
\Gamma & = \text{boundary of solution domain} \\
\Gamma_D & = \text{inflow boundary for SORTE} \\
\Gamma_N & = \text{outflow boundary for SORTE} \\
\Gamma_r & = \text{Dirichlet boundary for heat diffusion equation} \\
\Phi & = \text{scattering phase function} \\
\mu, \eta, \xi & = \text{direction cosine of radiation direction} \\
\kappa & = \text{absorption coefficient, m}^{-1} \\
\sigma & = \text{Stefan–Boltzmann constant, W/(m}^2\text{K}^4) \\
\sigma_S & = \text{Scattering coefficient, m}^{-1} \\
\tau & = \text{optical thickness, } \tau = \beta L \\
\alpha & = \text{temperature ratio, } \alpha = T_1/T_0 \\
\omega & = \text{scattering albedo} \\
\Omega, \Omega^+ & = \text{vector of radiation direction} \\
\Omega & = \text{solid angle} \\
\text{Subscripts} \\
i, j & = \text{spatial node index} \\
w & = \text{value at wall} \\
0 & = \text{value at } x=0 \\
L & = \text{value at } x=L \\
\text{Superscript} \\
m, m' & = \text{discrete ordinate direction} \\
\text{References} \\