Both Galerkin finite element method (GFEM) and least squares finite element method (LSFEM) are developed and their performances are compared for solving the radiative transfer equation of graded index medium in cylindrical coordinate system (RTEGC). The angular redistribution term of the RTEGC is discretized by finite difference approach and after angular discretization the RTEGC is formulated into a discrete-ordinates form, which is then discretized based on Galerkin or least squares finite element approach. To overcome the RTEGC-led numerical singularity at the origin of cylindrical coordinate system, a pole condition is proposed as a special mathematical boundary condition. Compared with the GFEM, the LSFEM has very good numerical properties and can effectively mitigate the nonphysical oscillation appeared in the GFEM solutions. Various problems of both axisymmetry and nonaxisymmetry, and with medium of uniform refractive index distribution or graded refractive index distribution are tested. The results show that both the finite element approaches have good accuracy to predict the radiative heat transfer in semitransparent graded index cylindrical medium, while the LSFEM has better numerical stability.
curved ray trajectories are also developed. Lemonnier and Le Dez [12] developed a discrete ordinates method to solve the radiative transfer in infinite slab with variable spatial refractive index. Liu [13] deduced the three-dimensional radiative transfer equation within graded index media in Cartesian coordinate system and developed a finite volume method (FVM) for solving multi-dimensional radiative transfer problems in graded index media. Following Ref. [13], Liu [14] developed a finite element method (FEM) for multi-dimensional graded index media. For radiative transfer in multi-dimensional graded index media, the curved ray tracing is very difficult and complicated. Hence the differential approaches which avoid the complicated computation of curved ray trajectories is more attractive in analyze practical system.

The most effective way to deal with cylindrical geometries is to map them to rectangular domains by using cylindrical coordinates, though the method formulated in Cartesian coordinates, such as the FVM and the FEM, are capable of solving radiative transfer in cylindrical refractive medium. As for radiative transfer in semitransparent graded index cylindrical medium, Ben Abdallah et al. [15] derived an integral form of radiative transfer equation inside refractive cylindrical media and solved the temperature distribution inside fibers. The basis procedure in the solution of the integral form equation is ray tracing which is computational complex and difficult. Recently, Liu et al. [16] derived the radiative transfer equation for graded index medium in cylindrical coordinate system, which can be used as basic equation to develop efficient differential approach to solve radiative transfer in semitransparent graded index cylindrical medium. To the authors’ knowledge, so far there is no differential approach developed in literature on radiative transfer in graded index cylindrical medium.

In this paper, based on the radiative transfer equation of graded index medium in cylindrical coordinate system (RTEGC) derived by Liu et al. [16], a discrete ordinates form of the RTEGC is derived and then two different finite element approach, namely, Galerkin finite element method (GFEM) and least squares finite element method (LSFEM), are developed and their performances are compared to solve the radiative transfer in semitransparent graded index cylindrical medium. Four various test problems of both axisymmetry and nonaxisymmetry, and with medium of uniform refractive index distribution or graded refractive index distribution are taken to verify the finite element formulation.

2. Mathematical formulation

2.1 Radiative transfer equation of graded index medium in cylindrical coordinate system

Consider radiative transfer in an enclosure filled with absorbing, emitting, and scattering gray medium with inhomogeneous and graded distribution of refractive index. As shown in Fig. 1, the radiative transfer equation in cylindrical coordinate system for an arbitrary medium can be written as

\[ \nabla \cdot \left( \kappa_s \nabla I \right) + \nabla \cdot \left( \kappa_a \nabla \phi \right) = \dot{S} \]

where \( \kappa_s \) is the scattering coefficient, \( \kappa_a \) is the absorption coefficient, \( \phi \) is the azimuthal angle step, \( \dot{S} \) is the source term.

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coordinate system can be written as [16]:

\[
\frac{\mu}{\rho} \frac{\partial I}{\partial \rho} + \frac{\eta}{\rho} \frac{\partial I}{\partial \theta} + \frac{\zeta}{\rho} \frac{\partial I}{\partial \phi} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[ \left( \mu \partial_x + \eta \partial_y + \zeta \partial_z \right) - \gamma \right] I(r, \Omega) + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \left[ \left( \beta \cos \phi - \mu \sin \phi \right) I(r, \Omega) \right] + [\kappa_a + \kappa_s] I(r, \Omega) = n^2 \kappa_a I_b(r) + \frac{K_s}{4\pi} \int l(r, \Omega') \phi(\Omega', \Omega) d\Omega' \quad (1)
\]

where

\[
\begin{align*}
\alpha &= \frac{1}{n} \frac{\partial n}{\partial \rho}, \\
\beta &= \frac{1}{n} \frac{\partial n}{\partial \theta}, \\
\gamma &= \frac{1}{n} \frac{\partial n}{\partial \phi}
\end{align*}
\]

(2a)

\[
\Omega = \mu \mathbf{e}_\rho + \eta \mathbf{e}_\theta + \zeta \mathbf{e}_\phi = \sin \theta \cos \phi \mathbf{e}_x + \sin \theta \sin \phi \mathbf{e}_y + \cos \theta \mathbf{e}_z
\]

(2b)

For the enclosure with opaque and diffuse walls, the boundary condition for Eq. (1) are given as

\[
l(r_w, \Omega) = c_\rho I_b(r_w) \frac{1 - \alpha_{\Omega,0}}{\pi} \int_{n_w > 0} l(r_w, \Omega') n_w \cdot \Omega' d\Omega', \quad \Omega \cdot n_w < 0
\]

(3)

Here \(l(r, \Omega)\) is the radiative intensity, which is a function of spatial position \(r\) and direction \(\Omega\); \(I_b(r)\) is the blackbody radiative intensity at the temperature of the medium; \(n\) is the refractive index of medium, which is a function of spatial position; \(\kappa_a\) and \(\kappa_s\) are the absorption and the scattering coefficients, respectively; \(\phi(\Omega', \Omega)\) is the scattering phase function from the incoming direction \(\Omega'\) to the outgoing direction \(\Omega\); \(\mu, \eta, \zeta\) are the direction cosine of the tangent vector of ray trajectory along the local orthonormal coordinates, \(\theta\) and \(\phi\) are the polar and the azimuthal angle, respectively; \(c_\rho\) is the wall emissivity, and \(n_w\) is the unit normal vector of the wall. The definitions of the spatial and angular variables in the cylindrical system are shown in Fig. 1.

Some basic numerical property of the RTEGC [Eq. (1)] can be easily explored from the form of the equation. By taking direction cosines \(\mu, \eta, \zeta\) as velocity in the \(\mathbf{e}_\rho\)-, \(\mathbf{e}_\theta\)- and \(\mathbf{e}_\phi\)-directions, respectively, the RTEGC can be considered as a special kind of convection-diffusion equation with convection-dominated characteristics for the vanishing of diffusion term. This is also a fact for the radiative transfer equation for uniform index media and in Cartesian coordinates [17,18]. It is well known that convection-diffusion equations are some of the most difficult problems to solve numerically, especially for convection dominated problem. The presence of convection term may cause nonphysical oscillatory of solutions. This type of instability can occur in many numerical methods including finite difference method and finite element method if no special stability treatment is taken. As such, the stability issue of different schemes of FEM for solving the RTEGC will be studied in following section.

Furthermore, it is noted that the term \(1/\rho\) in Eq. (1) will lead to numerical singularity at the origin where \(\rho = 0\). In this circumstance, special treatment and additional boundary conditions are needed in numerical discretization, which will be discussed in following section.
2.2. Pole condition and boundary conditions

To avoid the numerical singularity at origin $r = 0$, a simple approach is to multiply Eq. (1) by $r$ and yields

$$
\begin{align*}
\mu r \frac{\partial I(r, \Omega)}{\partial r} + \mu I(r, \Omega) + \eta \frac{\partial I(r, \Omega)}{\partial \psi} + \varsigma \frac{\partial I(r, \Omega)}{\partial z} - \frac{\partial \eta I(r, \Omega)}{\partial \phi} \\
+ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \left[ \xi (\mu \tilde{z} + \eta \tilde{\beta} + \varsigma \tilde{\gamma}) - \tilde{\eta} \right] I(r, \Omega) \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \left[ \tilde{\eta} \cos \phi - \tilde{z} \sin \phi \right] I(r, \Omega) \\
+ \rho [\kappa_a + \kappa_s] I(r, \Omega) = \rho n^2 \kappa_a b_0(r) + \rho \kappa_s \frac{\int l(r, \Omega') \Phi(\Omega', \Omega) d\Omega'}{4\pi} \tag{4}
\end{align*}
$$

where $\tilde{x} = \rho z = (\rho / n)(\partial n / \partial \rho)$, $\tilde{\beta} = \rho \beta = (1 / n)(\partial n / \partial \psi)$, $\tilde{\gamma} = \rho \gamma = (\rho / n)(\partial n / \partial z)$. Though Eq. (4) avoids the numerical singularity, it is still not ready for solving without appropriate boundary conditions.

As shown in Fig. 2, the physical domain of a circular enclosure is mapped to a square computational domain in cylindrical coordinates $(r, \Psi)$. In this square computational domain, besides the mapped original physical wall $(r = R)$, additional three mathematical boundaries $(r = 0, \Psi = 0, \Psi = 2\pi)$ appear and which need consistent boundary conditions.

For the top and bottom boundaries in the computational domain, namely $\Psi = 0$ and $\Psi = 2\pi$, respectively, it is easy to know that they are the same line in physical domain, hence they share the same radiative intensity on corresponding radius, and the constraint for this to mathematical boundaries is a periodical type boundary condition:

$$
l(r, \Omega)|_{\Psi = 0} = l(r, \Omega)|_{\Psi = 2\pi} \tag{5}
$$

![Fig. 2. Schematic of map from Cartesian coordinate to cylindrical coordinate and prescription of boundary conditions in the cylindrical coordinate system.](image-url)
As for the left boundary, namely \( \rho = 0 \), which is mapped from the origin of the physical disk, a constraint can be derived from Eq. (4). By setting \( \rho \to 0 \), Eq. (4) leads to

\[
\left[ \frac{\partial (r, \Omega)}{\partial \psi} - \frac{\partial (r, \Omega)}{\partial \phi} \right]_{\rho=0} = 0
\]

which is a mathematical boundary condition given for \( \rho = 0 \), or called pole condition. All the boundary conditions for the four sides of the square computational domain are depicted in Fig. 2. When \( \Omega \cdot n_w < 0 \), the physical boundary condition (Eq. (3)) is given to the right boundary; when \( \Omega \cdot n_w \geq 0 \), the mathematical boundary condition [Eq. (6)] is given to the left boundary.

2.3. Discrete ordinates form of the RTEGC

Eq. (1) differs from the radiative transfer equation in uniform index media, which contains three angular redistribution terms [the 4th, 5th and 6th term in Eq. (1)], the first one is due to the connection of the definition of angular coordinates with spatial coordinates, and the others account for the effect of curved ray trajectory due to graded index of refraction. Here, the piecewise constant angular approximation (PCA) in Ref. [14] is taken to discretize these angular redistribution terms, and the discrete polar and azimuthal angle are given as follows:

\[
\theta^m = (m - 1/2) \Delta \theta, \quad m = 1, 2, \ldots, N_\theta
\]

\[
\phi^n = (n - 1/2) \Delta \phi, \quad n = 1, 2, \ldots, N_\phi
\]

where \( \Delta \theta = \pi/N_\theta \) and \( \Delta \phi = 2\pi/N_\phi \) are the steps for the discretization of polar and azimuthal angles, respectively, \( N_\theta \) and \( N_\phi \) correspondingly denote the numbers of divisions. For each discrete direction \( \Omega^{m,n} \), the corresponding weight is

\[
w_\theta^n = \cos((\theta^m + \theta^m)/2) - \cos((\theta^m + \theta^{m+1})/2)
\]

\[
w_\phi^m = (\phi^{n+1} - \phi^{n-1})/2
\]

Following similar procedure presented in Ref. [14], finite difference discretization of the three redistribution terms in Eq. (4) can be written as

\[
- \frac{\partial \eta(r, \Omega^{m,n})}{\partial \phi} \approx \frac{\delta \eta}{\delta \phi} \bigg|_{\Omega^{m,n}} = \frac{X_{\theta}^{m,n+1/2} X_{\phi}^{m+1/2} - X_{\theta}^{m,n-1/2} X_{\phi}^{m-1/2}}{w_{\phi}^m}
\]

\[
\left\{ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[ (\bar{\xi} \mu \hat{\xi} + \eta \hat{\eta} + \zeta \hat{\zeta}) - \bar{\eta} \right] \right\}_{\Omega^{m,n}} \approx \frac{X_{\theta}^{m+1/2,n} X_{\phi}^{m+1/2} - X_{\theta}^{m-1/2,n} X_{\phi}^{m-1/2}}{w_{\phi}^m}
\]

\[
\left\{ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[ \beta \cos \phi - \bar{\lambda} \sin \phi \right] \right\}_{\Omega^{m,n}} \approx \frac{X_{\theta}^{m+1/2,n} X_{\phi}^{m+1/2} - X_{\theta}^{m-1/2,n} X_{\phi}^{m-1/2}}{w_{\phi}^m}
\]

and the recursion formula for \( X_{\theta}^{m+1/2,n}, X_{\phi}^{m,n+1/2} \) and \( X_{\phi}^{m,n+1/2} \) are obtained as following:

\[
X_{\theta}^{m+1/2,n} - X_{\theta}^{m-1/2,n} = -w_{\phi}^m \left[ \frac{\partial \eta}{\partial \phi} \right]_{\Omega^{m,n}} = -w_{\phi}^m \sin \theta^m \cos \phi^n
\]

\[
X_{\phi}^{m,n+1/2} = X_{\phi}^{m,N_\phi+1/2} = 0
\]

\[
X_{\theta}^{m+1/2,n} - X_{\theta}^{m-1/2,n} = \frac{w_{\phi}^m}{\sin \theta^m} \left[ \frac{\partial \beta}{\partial \phi} \cos \phi - \bar{\lambda} \sin \phi \right]_{\Omega^{m,n}} = \frac{w_{\phi}^m}{\sin \theta^m} (\cos 2\theta^m \cos \phi^n \bar{\lambda} + \cos 2\theta^m \sin \phi^n \bar{\lambda} - \sin 2\theta^m \bar{\zeta})
\]

\[
X_{\theta}^{m,n+1/2} = X_{\theta}^{m+1/2,n} = 0
\]

\[
X_{\phi}^{m,n+1/2} = X_{\phi}^{m+1/2,n-1/2} = \frac{w_{\phi}^m}{\sin \theta^m} \left[ \frac{\partial \lambda}{\partial \phi} \cos \phi - \bar{\lambda} \sin \phi \right]_{\Omega^{m,n}} = -w_{\phi}^m 1/\sin \theta^m (\cos \phi^n \bar{\lambda} + \sin \phi^n \bar{\zeta})
\]

\[
X_{\phi}^{m,n+1/2} = X_{\phi}^{m+1/2,n-1/2} = -\bar{\beta} \frac{1}{\sin \theta^m}
\]
Relations between the variables with fraction indices and the variables with integer indices are needed to close the discretization. The step scheme is used to close the discretization which set the downstream surface intensities equal to the upstream center intensities, namely,

\[ I_{m,n+1/2}^{m,n+1/2} = \max(I_{m,n+1/2}^{m,n}, 0) - \max(-I_{m,n+1/2}^{m,n+1/2}, 0) I_{m,n+1}^{m,n} \]  

(15a)

\[ I_{m,n-1/2}^{m,n-1/2} = \max(I_{m,n-1/2}^{m,n}, 0) - \max(-I_{m,n-1/2}^{m,n}, 0) I_{m,n}^{m,n} \]  

(15b)

\[ I_{m,n+1/2,n-1/2}^{m,n+1/2,n-1/2} = \max(I_{m,n+1/2,n-1/2}^{m,n}, 0) - \max(-I_{m,n+1/2,n-1/2}^{m,n}, 0) I_{m,n+1}^{m,n+1} \]  

(16a)

\[ I_{m,n-1/2,n+1/2}^{m,n-1/2,n+1/2} = \max(I_{m,n-1/2,n+1/2}^{m,n}, 0) - \max(-I_{m,n-1/2,n+1/2}^{m,n}, 0) I_{m,n-1}^{m,n} \]  

(16b)

\[ I_{m+1/2,n}^{m+1/2,n} = \max(I_{m+1/2,n}^{m,n+1/2}, 0) - \max(-I_{m+1/2,n}^{m,n+1/2}, 0) I_{m+1}^{m,n+1} \]  

(17a)

\[ I_{m,n-1/2}^{m,n-1/2} = \max(I_{m,n-1/2}^{m,n}, 0) - \max(-I_{m,n-1/2}^{m,n}, 0) I_{m,n}^{m,n} \]  

(17b)

By using the closure relations given in Eqs. (15)–(17), the discrete ordinate form of the RTEGC is obtained as

\[ \mu_{m,n}^{\phi} \frac{\partial I_{m,n}^{\phi}}{\partial r} + \eta_{m,n} \frac{\partial I_{m,n}^{\phi}}{\partial \psi} + \xi_{m,n} \frac{\partial I_{m,n}^{\phi}}{\partial z} + \tilde{\beta}_{m,n}^{\phi} I_{m,n}^{\phi} = S_{m,n}^{\phi} \]  

(18)

where \( \tilde{\beta}_{m,n}^{\phi} \) and \( S_{m,n}^{\phi} \) are functions of spatial coordinates and defined, respectively, as

\[ \tilde{\beta}_{m,n}^{\phi}(\mathbf{r}) = \frac{1}{w_{m,n}^{\phi}} \max(\chi_{m,n+1/2}^{m+1/2,n, 0}, 0) + \frac{1}{w_{m,n}^{\phi}} \max(-\chi_{m,n+1/2}^{m+1/2,n, 0}, 0) + \frac{1}{w_{m,n}^{\phi}} \max(\chi_{m,n+1/2}^{m,n+1/2}, 0) + \frac{1}{w_{m,n}^{\phi}} \max(-\chi_{m,n+1/2}^{m,n+1/2}, 0) \]

(19a)

\[ S_{m,n}^{\phi}(\mathbf{r}) = \rho m^2 \kappa_a b_0 + \frac{K_s}{4\pi} \sum_{m=1}^{N_\phi} \sum_{n=1}^{N_\psi} \sum_{m'=1}^{N_\phi} \sum_{n'=1}^{N_\psi} \frac{I_{m,n}^{m,n} \phi_{m,n}^{m,n} \psi_{m,n}^{m,n}}{w_{m,n}^{\phi} w_{m,n}^{\psi}} \]

(19b)

By the similar logic in analyzing the RTEGC [Eq. (1)], it can be seen that the discrete ordinate form of the RTEGC [Eq. (18)] is also in a form as a special kind of convection–diffusion equation with convection-dominated characteristics, and which will bring stability issue for the FEM discretization.

In case of infinite cylindrical medium with \( \partial I/\partial z = 0 \) and \( \partial n/\partial z = 0 \), Eq. (18) is simplified to two-dimension as

\[ \mu_{m,n}^{\phi} \frac{\partial I_{m,n}^{\phi}}{\partial r} + \eta_{m,n} \frac{\partial I_{m,n}^{\phi}}{\partial \psi} + \tilde{\beta}_{m,n}^{\phi}(\mathbf{r}) I_{m,n}^{\phi} = S_{m,n}^{\phi}(\mathbf{r}) \]  

(20)

Furthermore, if the problem is axisymmetric, namely, \( \partial I/\partial \psi = 0 \), Eq. (18) is simplified to one-dimension as

\[ \mu_{m,n}^{\phi} \frac{\partial I_{m,n}^{\phi}}{\partial r} + \tilde{\beta}_{m,n}^{\phi}(\mathbf{r}) I_{m,n}^{\phi} = S_{m,n}^{\phi}(\mathbf{r}) \]  

(21)

The discrete ordinates form of the mathematical boundary condition given for \( \rho = 0 \) (or the pole condition [Eq. (6)]) can be obtained similarly as

\[ \frac{\partial I_{m,n}}{\partial \psi} + \frac{I_{m,n+1}}{w_{m,n}} = \frac{I_{m,n+1}}{w_{m,n+1}} \]  

(22)

2.4. Discretization and implementation

Finite element approach is applied to discretize the discrete ordinate form of the RTEGC [Eq. (18)]. The computational domain (Fig. 2) is subdivided into many nonoverlaped elements, and shape functions \( (\phi_i) \) are then defined over these elements. The radiation intensity of direction \( \Omega_{m,n}^{\phi} \) can be approximated by the shape
functions as
\[ \rho_{i} = \sum_{i=1}^{N_{e}} I_{i} \rho_{i}(r) \quad \text{and} \quad I_{i}^{m,n} = \sum_{i=1}^{N_{e}} \rho_{i} I_{i} \phi_{i}(r) \] (23)
where \( \phi_{i} \) is the shape function of node \( i \), and \( I_{i}^{m,n} \) is the radiative intensity of node \( i \).

Substituting the finite element approximation [Eq. (23)] into Eq. (18), multiplying with weight function \( W_{j} \) and integration over the computational domain yields
\[ \sum_{i=1}^{N_{e}} I_{i}^{m,n} \int_{V} \left( \mu_{m,n} \frac{\partial I_{i}}{\partial \rho} + \mu_{n,m} \frac{\partial I_{i}}{\partial \psi} + \xi_{m,n} \rho_{i} \frac{\partial I_{i}}{\partial z} + \bar{\beta}_{i}^{m,n} \phi_{i} \right) W_{j} \, dV = \int_{V} S_{m,n}^{n} \, dV \] (24)

The conventional Galerkin finite element method is obtained when the weight functions \( W_{j} \) are taken as \( W_{j} = \phi_{j} \). As for least squares approach, the weight functions \( W_{j} \) are selected as \( W_{j} = \mu_{m,n} \rho_{i} \left( \frac{\partial \phi_{j}}{\partial \rho} + \eta_{m,n} \left( \frac{\partial \phi_{j}}{\partial \psi} \right) + \xi_{m,n} \rho_{i} \left( \frac{\partial \phi_{j}}{\partial z} \right) + \bar{\beta}_{i}^{m,n} \phi_{i} \right) \). For each discrete direction \( \Omega \), the final discretized radiative transfer equation can be written in the following set of linear equations:
\[ K^{m,n} X^{m,n} = H^{m,n} \] (25)
where the stiff matrix \( K^{m,n} \) and the right hand side vector \( H^{m,n} \) are defined as
\[ K_{ij}^{m,n} = \mu_{m,n} \rho_{i} \int_{V} \left( \frac{\partial \phi_{j}}{\partial \rho} \phi_{i} \right) dV + \mu_{n,m} \rho_{i} \int_{V} \left( \frac{\partial \phi_{j}}{\partial \psi} \phi_{i} \right) dV + \xi_{m,n} \rho_{i} \int_{V} \left( \frac{\partial \phi_{j}}{\partial z} \phi_{i} \right) dV + \bar{\beta}_{i}^{m,n} \int_{V} \phi_{j} \phi_{i} dV \] (26a)
\[ H_{j}^{m,n} = \int_{V} S_{m,n}^{n} \phi_{j} dV \] (26b)

For the least square approach, the stiff matrix \( K^{m,n} \) and the right hand side vector \( H^{m,n} \) are defined, respectively, as
\[ K_{ij}^{m,n} = \left( \mu_{m,n} \right) \rho_{i} \rho_{j} \int_{V} \left( \frac{\partial \phi_{j}}{\partial \rho} \phi_{i} \right) dV + \mu_{n,m} \rho_{i} \rho_{j} \int_{V} \left( \frac{\partial \phi_{j}}{\partial \psi} \phi_{i} \right) dV + \xi_{m,n} \rho_{i} \rho_{j} \int_{V} \left( \frac{\partial \phi_{j}}{\partial z} \phi_{i} \right) dV + \bar{\beta}_{i}^{m,n} \rho_{j} \int_{V} \phi_{j} \phi_{i} dV \] (27a)
\[ H_{j}^{m,n} = \mu_{m,n} \rho_{j} \int_{V} S_{m,n}^{n} \frac{\partial \phi_{j}}{\partial \rho} \phi_{i} dV + \mu_{n,m} \rho_{j} \int_{V} S_{m,n}^{n} \frac{\partial \phi_{j}}{\partial \psi} \phi_{i} dV + \xi_{m,n} \rho_{j} \int_{V} S_{m,n}^{n} \frac{\partial \phi_{j}}{\partial z} \phi_{i} dV + \bar{\beta}_{i}^{m,n} \rho_{j} \int_{V} S_{m,n}^{n} \phi_{j} \phi_{i} dV \] (27b)

The volume integration terms in Eq. (27) can be efficiently computed following the approach in Ref. [19]. The linear equation given in Eq. (25) is solved using Gaussian elimination. The boundary conditions are imposed by collocation approach as in Ref. [19]. The global iteration used in DOM is taken to update the source term.

### 3. Results and discussion

One- and two-dimensional computer codes have been developed based on the finite element formulation described above. Four various problems of both axisymmetry and nonaxisymmetry, and with medium of uniform refractive index distribution or graded refractive index distribution are taken to verify the performance of the presented GFEM and LSFEM. The radius of the cylindrical enclosure is taken as \( R = 1 \) m. The absorption and the scattering coefficients of the medium enclosed by the cylindrical enclosure are uniform. For the sake of comparison, the relative error based on data in the references is defined as:

\[ \text{Relative error\%} = \frac{\int |\text{FEM solution}(\rho) - \text{Benchmark result}(\rho)| \, d\rho}{\int |\text{Benchmark result}(\rho)| \, d\rho} \times 100 \] (28)
3.1. Case 1: cylindrical medium with parabolic refractive index distribution

We consider radiative transfer problem in a one-dimensional nonscattering semitransparent cylinder bounded by black walls. The temperature of cylinder wall is $T_w = 1000\,\text{K}$. The refractive index within the cylinder has a parabolic profile with the radius as $n(\rho) = \sqrt{n_0^2 - b^2 (\rho/R)^2}$, in which $n_0 = 2$ and $b = 1$. For this distribution of refractive index, the ray trajectory can be obtained analytically [20], hence a ray tracing method following Liu [11] is easily developed to obtain the intensity distribution. The problem of this case is axisymmetrical. The GFEM is applied to this case with 30 isoparametric linear elements and the solid angle is subdivided as $N_0 = 20$. Fig. 3(a) and (b) present the radial dimensionless net radiative heat flux distributions and the dimensionless incident radiation along the radial coordinate, respectively, for three different values of absorption coefficient, namely, $\kappa_a = 0.5, 1.0$ and 10, and compared with the solutions obtained by ray tracing method. It can be seen that the results of GFEM are in good agreement with the results obtained by ray tracing method for different values of optical thickness. The maximum relative error based on the data of ray-tracing method is less than 3.2%.

3.2. Case 2: cylindrical medium with Gaussian shaped radiative source term

We consider radiative transfer in a nonscattering medium within an infinite black cylinder enclosure. The temperature of the medium is $T_g = 1000\,\text{K}$. The optical thickness based on the radius of the cylinder is $\tau_R = \kappa_a R = 1.0$. The radiative source term of the medium has a profile similar to a Gaussian function. This problem is modeled by the following RTE as

\[
\frac{\partial \phi}{\partial \rho} - \frac{\partial \phi}{\partial \theta} \left[ \frac{\partial \phi}{\partial \theta} \right] + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \left[ \frac{\partial \phi}{\partial \phi} \right] - \kappa_a \phi (\rho, \Omega) = \frac{\sigma T^4}{\pi} e^{-\rho c^2 z^2}, \quad \rho, c \in [0, 1]
\]

in which the source term takes Gaussian shaped form. Both the GFEM and the LSFEM are applied to the case of $c = 0.5$ and $z = 0.06$. As a simple example, the medium with uniform refractive index is considered. Fig. 4 presents the obtained radial dimensionless net radiative heat flux distributions and the dimensionless incident along radial coordinate $\rho$ for the case of medium with uniform refractive index, respectively, by the GFEM and the LSFEM. Same spatial and angular grid are used by the GFEM and the LSFEM, namely, 20 isoparametric linear elements and the solid angle is subdivided as $N_0 = 200$. Here, the results obtained by LSFEM with 100 linear elements and $N_0 = 200$ are taken as benchmarks. A single peak is observed in the curve of incident radiation at $\rho/R = 0.5$, which results from the highest emission given by Gaussian shape source term caused at point of $\rho/R = 0.5$. It can be seen that both the radiative heat flux and the dimensionless incident radiation obtained by the GFEM exhibit obvious nonphysical oscillations, which is due to the instability caused by the convection-dominated property of the RTEGC as discussed previously. Similar kind of nonphysical oscillations of results has also been report elsewhere for the method based on radiative transfer equation in Cartesian coordinate system [18]. However, the results obtained by the LSFEM are free of the nonphysical ‘wiggles’ which demonstrate the LSFEM has better numerical properties.

To further demonstrate the performance of the GFEM and the LSFEM for solving radiative transfer in cylindrical medium with graded refractive index, Fig. 5 presents the solved dimensionless net radiative heat flux distributions along radial coordinate $\rho$, respectively, by the GFEM and the LSFEM for two different types of refractive index...
distribution: (1) \( n(\rho) = 1 + 2\rho/R \) and (2) \( n(\rho) = 5\sqrt{1 - (0.66\rho/R)^2} \). The same spatial and angular grids are used as in the uniform refractive index case. It can be seen that the dimensionless net radiative heat flux distributions and the incident radiation obtained by the GFEM exhibit nonphysical oscillations, while the LSFEM still gives stable results. As such, the LSFEM has better numerical stability is demonstrated.

### 3.3. Case 3: isotropic scattering cylinder with transparent boundary

An isotropic scattering medium enclosed by transparent cylindrical boundary is considered. The temperature of surrounding and the medium are kept \( T_w = 1000 \) K and \( T_g = 0 \) K, respectively. The dimensionless incident radiation with the optical radius of 1.0 is solved by the LSFEM for four different single scattering albedo, namely, \( \omega = 0.3, 0.5, 0.7 \) and 0.9. The results are presented in Table 1 and compared with the results from Refs. [21,22]. The LSFEM is applied here with 100 uniformly subdivided isoparametric linear elements and the total solid angle is subdivided as \( N_y/N_j = 40/80 \). We note that the curve of incident radiation becomes more flat for larger values of scattering albedo (i.e., \( \omega = 0.9 \)). As shown in Table 1, the results obtained by using the LSFEM are in excellent agreement with those obtained from the Refs. [21,22].

For the sake of finding out the influence of refractive index on radiative heat transfer property, as shown in Fig. 6, the dimensionless incident radiation and the dimensionless net radiative heat flux are obtained by the LSFEM, respectively, under the condition of three different refractive index distributions: (1) uniform index \( n(\rho) = 2.5 \), (2) linear graded index \( n(\rho) = 1 + 2\rho/R \) and (3) parabolically graded index \( n(\rho) = \sqrt{4 - (\rho/R)^2} \). Here, the radial direction is subdivided to 400...
isotropically incident external irradiation on a solid cylinder having a transparent boundary surface.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$G(0)/4\pi T_w^4$</th>
<th>$G(R)/4\pi T_w^4$</th>
<th>$G(R)/4\pi T_w^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Refs.</td>
<td>This paper</td>
<td>Refs.</td>
</tr>
<tr>
<td>0.3</td>
<td>0.36436 [21]</td>
<td>0.37423</td>
<td>0.41631 [21]</td>
</tr>
<tr>
<td>0.5</td>
<td>0.45701 [21]</td>
<td>0.46543</td>
<td>0.50592 [21]</td>
</tr>
<tr>
<td>0.7</td>
<td>0.59661 [22]</td>
<td>0.59656</td>
<td>–</td>
</tr>
<tr>
<td>0.9</td>
<td>0.82466 [21]</td>
<td>0.82807</td>
<td>0.84417 [21]</td>
</tr>
</tbody>
</table>

![Fig. 6.](image)

(a) Dimensionless incident radiation and (b) dimensionless net radiative heat flux distributions with transparent boundary.

isoparametric linear elements and the total solid angle is subdivided as $N_x \times N_y = 20 \times 40$. The single scattering albedo is $\omega = 0.3$. With the distribution of refractive index varies from uniform index to linear graded index and parabolically graded index, the dimensionless incident radiation and the dimensionless net radiative heat flux inside the cylinder varies obviously. This implicates that with properly predefined distribution of refractive index, the distribution characteristics of radiation field in the medium may be controlled to meet the special demand of engineering application.

3.4. Case 4: partially heated cylindrical medium with graded index of refraction

To further check the validity and performance of the finite element formulation for radiative transfer in two-dimensional graded index cylindrical medium, as shown in Fig. 7, we consider a cylindrical medium is partially heated at the bottom half wall, which will result in a nonaxisymmetric radiation field. The temperature of the bottom half wall is $T_{w2} = 1000$ K and the upper half wall is kept cold ($T_{w1} = 0$ K). The medium is isotropically scattering and with a parabolically graded index distribution as studied in Case 1. The cylinder wall is black. The LSFEM is applied to this case with 400 bilinear quadrilateral elements and the total solid angle is subdivided as $N_x \times N_y = 20 \times 40$. Fig. 8 presents the distributions of the dimensionless incident radiation and the dimensionless net radiative heat flux along the radial direction of $\Psi = 3\pi/2$ and $\Psi = \pi/2$ (or $y$ axis in the corresponding Cartesian coordinate system) for three different values of single scattering albedo, namely, $\omega = 0.0, 0.5$ and 1.0. For the sake of clearly presenting the results, the radial coordinate $\rho$ is assigned a minus sign for $\Psi = 3\pi/2$. The results for the case of uniform refractive index distribution are also presented to understand the effect of graded index distribution. Here, the results obtained using a calibrated Cartesian FEM code [14] are also presented and compared with solutions obtained by the LSFEM developed in this paper. The results of the Cartesian FEM code are obtained with 590 linear triangular elements and the total solid angle is subdivided as $N_x \times N_y = 20 \times 40$ parts. For different strength of scattering, the trends of incident radiation and radial radiative heat flux distribution show similar pattern for both uniform and graded index distribution. It can be seen that the results obtained by the cylindrical coordinates based LSFEM developed in this paper are in good agreement with the GFEM solution for different values of scattering albedo. The LSFEM can be used to effectively solve radiative transfer in multi-dimensional absorbing, emitting, and scattering cylindrical medium with graded index distribution.

4. Conclusions

A discrete ordinates form of the radiative transfer in graded index medium for cylindrical coordinate system is derived. Both the original and the discrete ordinates form of the radiative transfer in graded index medium for cylindrical
coordinate system is in a form as a special kind of convection–diffusion equation of convection-dominated characteristics, and which will bring stability issue for the FEM discretization. Two different finite element approaches, namely, Galerkin finite element method and least squares finite element method are developed and their performances are compared to solve the radiative transfer in semitransparent graded index cylindrical medium. A pole condition is proposed as a special mathematical boundary condition. Four various test problems are taken to verify the validity and performance of the finite element approaches. Generally, both the finite element approaches has good accuracy to predict the radiative heat transfer in semitransparent graded index cylindrical medium, while the LSFEM has better numerical stability which can effectively mitigate the nonphysical oscillation appeared in the GFEM solutions.

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References

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