User’s guide of European options and American options

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1 Pricing options

In the mathematical finance, we know that European options and American option can be described by linear BSDE and linear reflected BSDE, respectively. Let \(x_t\) be the price of the underlying stock which is a geometric Brownian motion:

\[
x_t = x_0 + \int_0^t (r + \sigma \theta) x_s ds + \int_0^t \sigma x_s dB_s.
\]

Here \(r\) is interest rate, \(\sigma\) is volatility, \(\theta\) is risk premium. In following, \(R\) is noted as borrow rate, in many cases, we assume that \(R = r\). Here we consider also the cases of \(R \neq r\).

For European call option, the wealth \(y_t\) satisfies the following BSDE

\[
y_t = (x_T - k)^+ - \int_t^T (ry_s + \sigma \theta z_s) ds + \int_t^T (R - r)(y_s - z_s)^- ds - \int_t^T \sigma z_s dB_s, 0 \leq t \leq T.
\]

For European put option, the wealth \(y_t\) satisfies the following BSDE

\[
y_t = (k - x_T)^+ - \int_t^T (ry_s + \sigma \theta z_s) ds + \int_t^T (R - r)(y_s - z_s)^- ds - \int_t^T \sigma z_s dB_s, 0 \leq t \leq T.
\]

For the American call option, the wealth \(y_t\) satisfies the following equation

\[
y_t = (x_T - k)^+ - \int_t^T [ry_s + \sigma \theta z_s] ds + \int_t^T (R - r)(y_s - z_s)^- ds + A_T - A_t - \int_t^T \sigma z_s dB_s, 0 \leq t \leq T, dA \leq 0,
\]

and

\[
y_t \geq S_t, \text{ for } 0 \leq t \leq \tau, \text{ where } S_t = (x_t - k)^+, \int_0^T (y_t - S_t) dA_t = 0,
\]

where \(\tau\) is a stopping time, and \(\tau = \inf\{t, y_t - S_t < 0\}\). In finance, \(y_t\) is wealth process, \(\tau\) stand for exit time from market for the investor. At this time he will do the action buying or selling the stock. In this problem, we are interested in the \(y_t, z_t, \text{ and } \tau\). To solve it, we consider it as a reflected BSDE, with the barrier \(S_t\), then \(\tau = \inf\{t, A_t > 0\}\).

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For American put option, the investor’s wealth \( y_t \) satisfies the RBSDE:

\[
y_t = (k - x_T)^+ - \int_t^T [ry_s + \sigma \theta z_s] ds + \int_t^T (R - r)(y_s - z_s)^- ds + A_T - A_t - \int_t^T z_s dB_s, 0 \leq t \leq 1
\]

\[
y_t \geq S_t, dA_t \geq 0, \int_0^T (y_t - S_t) dA_t = 0,
\]

where \( S_t = (k - x_t)^+ \). At the stopping time \( \tau = \inf \{t, A_t > 0\} \), the investor will execute the contract and his return is \( (k - x_\tau)^+ \).

1.1 User’s guide for Pricing options.

The programs are realized by Matlab’s *.p files. To run these programs, Matlab 5.3 or higher version is required. The present package is compressed as "priceoption.zip". After download the compressed file, you should:

1. Uncompress the file “priceoption.zip” in the document C:\matlab\work (or D:\..., your Matlab is installed in the hard disk D:\).

2. Run the Matlab command window.

3. Then within this window single click “File” in the menu buttons and then, among the prompted file buttons, single click “Set Path” button. Then within the prompted “Path Browse” window, browse and add the direction C:\matlab\work\price option2 in the Path.

4. After these preparation, you can run our program: in Matlab’s command window. For pricing options, type “priceoption” followed with a “return”. Main interface is generated.

On the left-up portion, we list out the formula for pricing option. User may choose the different type option he wants to calculate by the menu on the right-top side. After choosing
the type of option by popmenu, the formulas on the left-up portion is changed at same time. For European put option, we will have

**European put option**

\[ -dy_t = \left( -r \cdot y_t + \sigma \cdot \theta \cdot \mathcal{N} \right) dt + \left( (R - r) \cdot (y_t - \mathcal{N}) - \sigma \cdot \mathcal{N} \right) dB_t \]

\[ y_T = (k - x_T)^+. \]

*Stock price:* \( dx_t = \left( r + \theta \right) \cdot x_t dt + \sigma \cdot x_t \cdot dB_t \), \( x(0) = x_0 \).

*Option price:* \( y_0 \).

![Figure 2: Interface for Pricing options](image)

For American call option, we will have

**American call option**

\[ -dy_t = \left( -r \cdot y_t + \sigma \cdot \theta \cdot \mathcal{N} \right) dt + \left( (R - r) \cdot (y_t - \mathcal{N}) - \sigma \cdot \mathcal{N} \right) dB_t \]

\[ y_T = (x_T - k)^+. \]

*Stock price:* \( dx_t = \left( r + \theta \right) \cdot x_t dt + \sigma \cdot x_t \cdot dB_t \), \( x(0) = x_0 \).

*Option price:* \( y_0 \).

![Figure 3: Interface for Pricing options](image)
For American put option, we will have

\[ -dy_t = -[r \cdot y_t + \sigma \cdot \theta \cdot z_t] dt + (R-r) (y_t - z_t)^- dt - dA_t - \sigma \cdot z_t dB_t \]
\[ y_T = (k - x_T)^+. \]

Stock price:
\[ dx_t = (r + \sigma \cdot \theta) \cdot x_t dt + \sigma \cdot x_t dB_t, \quad x(0) = x_0. \]
\[ S_t = (k - x_t)^+, \quad \text{and } Y_t \geq S_t, \quad 0 \leq t \leq \tau. \]

Option price = \( y_0 \).

Choose an option:

<table>
<thead>
<tr>
<th>Choose an option:</th>
</tr>
</thead>
<tbody>
<tr>
<td>American put option</td>
</tr>
</tbody>
</table>

Figure 4: Interface for Pricing options

On the left-lower side, you can input the parameters: \( r, R, \sigma, \theta, x_0, k, T, N \) (discretitation parameter). The default values are

\[ r = 0.03, R = 0.04, \theta = 0.1, \sigma = 0.3, \]
\[ x_0 = 100, k = 100, T = 1, N = 400. \]

After inputting the parameters, you can use these programs to do the calculation. User will see different simulation result by clicking the buttons on the left side. The functions of each button is similar for European call option, European put option, American call option and American put option. Here we take European call option for example to explain the functions of buttons.

1. Clicking the button 'calculate' on the left-side once, then the program of calculation will run. For getting the result, calculation will take certain seconds and then indicate the end of the calculation by jumping a dialogue-box "the calculation is complete."

2. Click the next button “progress”, the program will draw the backward computation procedure of the function \( y(t, x) \), dynamically and backwardly on the left-up subfigure. In the figure the red lines above (resp. blue lines below) show the solution \( y \) (resp. Brownian motion). (For American options, the grid surface is the barrier). At the end, we can see a vertical red line, which simplify the value of solution \( y \) at time 0. Then the above colorful surface of solution \( y \) is generated in same place, which is named ”surface for solution y”. The green lines in this figure show the relation between the solution \( y \) and the Brownian motion, while the blue line below indicate the range of discrete Brownian motion.

The next three buttons on the main user-interface are for the simulations of the solution.
3. Click the button “B.M. price & wealth”, a dynamically generated Brownian path will appear on the right-up subfigure, 'Sample way of Brownian Motion'. On the left-up subfigure, the green line on the bottom is price process corresponding to the Brownian motion sample, while the red line is the corresponding wealth line. These paths will terminated by a jump of a vertical or horizon line indicating the terminal value $y_T = \xi(\omega)$ of this sample. And the grey line on the bottom is the expected return. If you click the button “more” on this new figure, then another Brownian path and the related price process, wealth process will be produced in the same subfigure.

4. Click the button “wealth & delta”, you will see a moving $(x_t, y_t, \Delta_t, z_t)$ (or $(x_t, y_t, \Delta_t, z_t, A_t)$ for American option) on left-up subfigure. Each subfigure is noted on the top.

<table>
<thead>
<tr>
<th>the price $x_t$</th>
<th>wealth process $y_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>process $\Delta_t$</td>
<td>process $z_t$</td>
</tr>
</tbody>
</table>

The grey line in subfigure of price $x_t$ is expected return.

For American option, we will see

<table>
<thead>
<tr>
<th>the price $x_t$</th>
<th>wealth process $y_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>process $\Delta_t$</td>
<td>process $z_t$</td>
</tr>
</tbody>
</table>

The grey line in the subfigure of wealth process $y_t$ is the barrier $(x_t - k)^+ \text{ or } (k - x_t)^+$. On this figure and the following two, the grey point is for the exit time $\tau = \inf\{t, A_t > 0\}$.

Click the bottom 'more', another group of trajectories will be shown on the same subfigure.

5. Clicking the following button ”B.M. and solution y”, you will see the sample way of solution $y$ on the solution surface on left-up subfigure. On the above window of it, a trajectory of Brownian Motion $B_t(\omega)$ is showed on the ground, while the solution $y_t$ according to this Brownian motion is showed on the solution surface. For the case of American option, there is a grey surface is for the barrier $S_t = (x_t - k)^+ \text{ or } (k - x_t)^+$. And the free boundary is shown by a blue line. In additional subfigure, there are the trajectories of $A_t$ (on the left). The button ”more” is still for a new group of lines produced in the different color. And the exact exit is noted by a sentence.

Remark 1 This simulation is shown only when the discretation number $N = 400$. Otherwise there will come an error dialog notice.

6. Click the next button ”$R = r$ or $R > r$”, we will see a new figure.
European call option

\[ -dy_t = \left[ r \cdot y_t + \sigma \cdot \theta \cdot z_t \right] dt + \left( R - r \right) \left( y_t - z_t \right)^{-} \cdot dt - \sigma \cdot z_t dB_t \]

\[ y_T = (x_T - k)^+ \]

Stock price: \[ dx_t = \left( r + \sigma \cdot \theta \right) \cdot x_t dt + \sigma \cdot x_t dB_t \], \[ x(0) = x_0 \]

Option price: \[ y_0 \]

Choose an option:

- Input r:
- Input R:
- Input \( \theta \):
- Input \( \sigma \):
- Input \( x_0 \):
- Input k:
- Input T:
- Input discretation parameter:

Figure 5: Compare the case \( R = r \) with \( R > r \)

This is for user to compare the simulation results of solution of the case \( R = r \) and \( R > r \), with same parameters. In this case, the input of \( R \) and \( r \) must be different, otherwise an error dialog notice will be generated.

Then user should click the button ”calculate” let program calculate the solution of option with parameter \( R = r \) and \( R > r \). For example, if the inputs are \( R = 0.05 \), \( r = 0.03 \), then program will calculate the solution of option with parameter \( r = R = 0.03 \) and \( r = 0.03 \), \( R = 0.05 \). When calculation is done, a dialog-box will generate with notice ’the calculation is ready.’ Then user may the two buttons on left side to see simulation results.

Click the button ”B.M. price & wealth”, as item 3, program will generate dynamically a trajectory of Brownian motion and corresponding trajectory of price process (green line) and solution of \( R = r \) (blue line) and \( R > r \) (red line). If two trajectories of solutions are close, blue line may be covered by red line. Click button ”more”, a real group of trajectories will be generated in the same sub-figure.

Click the button ”wealth & delta”, as item 4, program will generate a group of trajectories of price \( x_t \), wealth \( y_t \), \( \Delta_t \), \( z_t \), with process \( A_t \) for American option. For last three or four simulations, there are two lines shown in same sub-figure, one darker for the case \( R > r \), the other is clearer for the case \( R = r \). Click button ”more”, a real group of trajectories will be generated in the same sub-figure.

Click the button ”return”, main interface will return to original one as shown in Figure 1-4, respectively.

7. For closing the figures there is two ways. One is using the button ”Close” on them. The other is to click the little cross on the right-up corner of the figure.