Exercises of Petersen's Riemannian Geometry

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January 14, 2010

To my parents, Jihe Zhang and Yulan Ouyang

Contents

1 Riemannian Metrics

1.1 On product manifolds $M \times N$ one has special product metrics $g = g_M + g_N$, where g_M, g_N are metrics on M, N respectively.

- Show that $(\mathbb{R}^n, can) = (\mathbb{R}, dt^2) \times \cdots \times (\mathbb{R}, dt^2)$.
- Show that the flat square torus

$$T^2 = \mathbb{R}^2/\mathbb{Z}^2 = \left(S^1, \left(\frac{1}{2\pi}\right)^2 d\theta^2\right) \times \left(S^1, \left(\frac{1}{2\pi}\right)^2 d\theta^2\right).$$

- Show that

$$F(\theta_1, \theta_2) = \frac{1}{2\pi}(\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2)$$

  is a Riemannian embedding: $T^2 \to \mathbb{R}^4$.

Proof. • $can_{\mathbb{R}^n} = \sum_{i=1}^n (dx^i)^2$.

- Note that

$$\left(\frac{1}{2\pi}\right) d(\theta_1)^2 + \left(\frac{1}{2\pi}\right) d(\theta_2)^2 = \left(d\left(\frac{\theta_1}{2\pi}\right)\right)^2 + \left(d\left(\frac{\theta_1}{2\pi}\right)\right)^2,$$

  where $\theta_1, \theta_2 \in [0, 2\pi)$.

- ★ F is injective.

$$\sin\theta_1 = \sin\theta_2, \cos\theta_1 = \cos\theta_2 \Rightarrow \theta = 0.$$

  ★ dF is injective.

$$dF(\partial_{\theta_1}) = \frac{1}{2\pi}(-\sin\theta_1, \cos\theta_1, 0, 0),$$

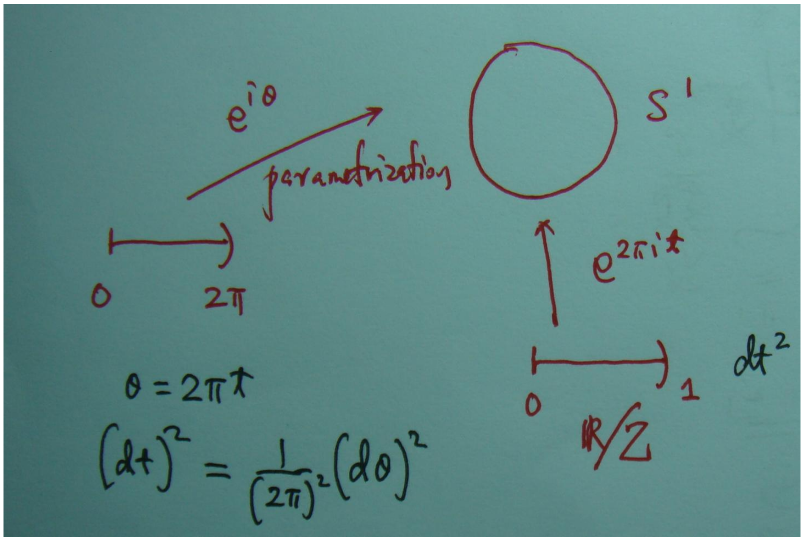

Figure 1: The isometry between T^1 and S^1 with metric above

$$dF(\partial_{\theta_2}) = \frac{1}{2\pi}(0, 0, -\sin\theta_2, \cos\theta_2).$$

$$dF(\alpha\partial_{\theta_1} + \beta\partial_{\theta_2}) = 0 \Rightarrow \alpha = 0 = \beta.$$

★ F is a Riemannian embedding.

Just note that

$$\begin{aligned} F^* can_{\mathbb{R}^4} &= \left(d\left(\frac{1}{2\pi}cos\theta_1\right)\right)^2 + \cdots \\ &= \left(\frac{1}{2\pi}\right)^2\left((d\theta_1)^2 + (d\theta_2)^2\right). \end{aligned}$$

$\square$

1.5 Let G be a Lie group.

- Show that G admits a bi-invariant metric, i.e. both right and left translation are isometries.

- Show that the inner automorphism $Ad_h(x) = hxh^{-1}$ is a Riemannian isometry. Conclude that its differential at $x = e$ denoted by the same letters

$$Ad_h : \mathfrak{g} \to \mathfrak{g}$$

  is a linear isometry with respect to g.

- Use this to show that the adjoint action

$$\begin{aligned} ad_U &: \mathfrak{g} \to \mathfrak{g}, \\ ad_U(X) &= [U, X] \end{aligned}$$

  is skew-symmetric, i.e.,

$$g([U, X], Y) = -g(X, [U, Y]).$$

Proof. • Let g_L be a left-invariant metric, i.e.

$$\begin{aligned} g_L(v, w) &= g_L((dL_{x^{-1}})_x(v), (dL_{x^{-1}})_x(w)), \\ &\forall\ v, w \in T_xG,\ x \in G. \end{aligned}$$

Let $E_1, \cdots, E_n$ the left-invariant orthonormal vector fields, $\sigma^1, \cdots, \sigma^n$ the dual 1-forms. Thus the volume form

$$\omega = \sigma^1 \wedge \cdots \wedge \sigma^n.$$

Define

$$g(v, w) = \frac{1}{\int \omega} \int g_L(DR_x(v), DR_x(w))\omega.$$

We have

★ g is left-invariant.

$$\begin{aligned}
& g((DL_y)_e(v),(DL_y)_e(w)) \\
= & \frac{1}{\int\omega}\int g_L((DR_x)_y\circ(DL_y)_e(v),(DR_x)_y\circ(DL_y)_e(w))\omega \\
= & \frac{1}{\int\omega}\int g_L((DL_y)_x\circ(DR_x)_e(v),(DL_y)_x\circ(DR_x)_e(w))\omega \\
& (R_x\circ L_y=L_y\circ R_x) \\
= & \frac{1}{\int\omega}\int g_L((DR_x)_e(v),(DR_x)_e(w))\omega \\
& (g_L \text{ is left-invariant}) \\
= & g(v,w),\ \forall\ v,w\in T_eG;\ y\in G.
\end{aligned}$$

★ g is right-invariant.

$$\begin{aligned}
& g((DR_y)_e(v),(DR_y)_e(w)) \\
= & \frac{1}{\int\omega}\int g_L((DR_x)_y\circ(DR_y)_e(v),(DR_x)_y\circ(DR_y)_e(w))\omega \\
= & \frac{1}{\int\omega}\int g_L((DR_{yx})_e(v),(DR_{yx})_e(w))\omega \\
= & \frac{1}{\int\omega}\int g_L((DR_z)_e(v),(DR_z)_e(w))\omega \\
& (\text{ change of variables: } z=yx\ ) \\
= & g(v,w),\ \forall\ v,w\in T_eG;\ y\in G.
\end{aligned}$$

- Indeed,

$$Ad_h=L_h\circ R_{h^{-1}},$$

thus its differential

$$D(Ad_h)=(DL_h)\circ(DR_{h^{-1}}),$$

and

$$\begin{aligned}
& g(Ad_h(v), Ad_h(w)) \\
= \ & g(D(Ad_h)_e(v), D(Ad_h)_e(w)) \\
= \ & g((D_h)_{xh^{-1}} \circ (DR_{h^{-1}})x(v), (D_h)_{xh^{-1}} \circ (DR_{h^{-1}})x(w)) \\
= \ & g(v, w), \ \forall \ v, w \in T_eM,
\end{aligned}$$

i.e. Ad_h is a linear isometry w.r.t. g.

- By the second assertion,

$$g(Ad_{exp(tU)}X, Ad_{exp(tU)}Y) = g(X, Y), \ \forall X, Y \in \mathfrak{g}.$$

Differentiating the above equality at $t = 0$, we get

$$g(ad_U X, Y) + g(X, ad_U Y) = 0,$$

i.e.

$$g([U, X], Y) = -g(X, [U, Y]).$$

□

1.6 Let V be a n-dimensional vector space with a symmetric nondegenerate bilinear form g of index p.

- Show that there exists a basis $e_1, \cdots, e_n$ such that $g(e_i, e_j) = 0$ if $i \neq j$, $g(e_i, e_i) = -1$ if $i = 1, \cdots, p$ and $g(e_i, e_i) = 1$ if $i = p+1, \cdots, n$. Thus V is isometric to $\mathbb{R}^{p,q}$.
- Show that for any v we have the expansion

$$v = \sum_{i=1}^{n} \frac{g(v, e_i)}{g(e_i, e_i)} e_i = -\sum_{i=1}^{p} g(v, e_i)e_i + \sum_{i=p+1}^{n} g(v, e_i)e_i.$$

- Let $L : V \to V$ be a linear operator. Show that

$$tr(L) = \sum_{i=1}^{n} \frac{g(L(e_i), e_i)}{g(e_i, e_i)}.$$

Proof. Indeed, nothing need to show if one is familiar with the theory of quadratic forms! □

2 Curvature

2.1 Show that the connection on Euclidean space is the only affine connection such that $\nabla X = 0$ for all constant vector fields X.

Proof. If $\nabla X = 0,\ \forall\ X = a^i \partial_i$, with a^i constant, then for $\forall j$,

$$\begin{aligned} 0 &= \nabla_{\partial_j} X = \nabla_{\partial_j}(a^i \partial_i) \\ &= (\partial_j a^i)\partial_i + a^i \nabla_{\partial_j} \partial_j = a^i \nabla_{\partial_j} \partial_i \\ &= a^i \Gamma_{ji}^k \partial_k, \end{aligned}$$

i.e.

$$\Gamma_{ij}^k = 0,\ \forall\ i, j, k,$$

the connection is flat. □

2.2 If $F : M \to M$ is a diffeomorphism, then the push-forward of a vector field is defined as

$$(F_* X)|_p = DF(X|_{F^{-1}(p)}).$$

Let F be a isometry on (M, g).

- Show that $F_*(\nabla_X Y) = \nabla_{F_* X} F_* Y$ for all vector fields.

- If $(M,g) = (\mathbb{R}, can)$, then isometries are of the form $F(x) = Ox + b$, where $O \in O(n)$ and $b \in \mathbb{R}^n$.

Proof.

- By Koszul formula, we have for $\forall Z \in \mathcal{X}(M)$,

$$\begin{aligned}
& g(F_*(\nabla_X Y), F_* Z) \circ F = (F^* g)(\nabla_X Y, Z) \\
= & \frac{1}{2}\left[X((F^*g)(Y,Z)) + Y((F^*g)(Z,X)) - Z((F^*g)(X,Y))\right. \\
& \left.+(F^*g)(Z,[X,Y]) + (F^*g)(Y,[Z,X]) - (F^*g)(X,[Y,Z])\right] \\
= & \frac{1}{2}\left[X(g(F_*Y, F_*Z) \circ F) + Y(g(F_*Z, F_*X) \circ F)\right. \\
& -Z(g(F_*X, F_*Y) \circ F) \\
& +g(F_*Z, [F_*X, F_*Y]) \circ F + g(F_*Y, [F_*F_*Z, F_*X]) \circ F \\
& \left.+g(F_*X, [F_*Y, F_*Z]) \circ F\right] \\
= & g(\nabla_{F_*X} F_*Y, F_*Z) \circ F.
\end{aligned}$$

- If $(M,g) = (\mathbb{R}^n, can)$, and F is an isometry, then due to

$$0 = F_*(\nabla_{\partial_i} \partial_j) = \nabla_{F_*\partial_i} F_* \partial_j,$$

we have

$$\nabla F_* \partial_j = 0.$$

Thus

$$\frac{\partial^2 F_i}{\partial x^j \partial x^k} = 0, \ \forall\ i, j, k.$$

While, F is an isometry implies

$$\begin{aligned}
\delta_{ij} &= \langle \partial_i, \partial_j \rangle \\
&= \langle F_* \partial_i, F_* \partial_j \rangle \\
&= \left\langle \frac{\partial F_i}{\partial x_k} \partial_k, \frac{\partial F_j}{\partial x_l} \partial_l \right\rangle \\
&= \sum_k \frac{\partial F_i}{\partial x_k} \frac{\partial F_j}{\partial x_k}.
\end{aligned}$$

i.e.

$$\left[\frac{\partial F_i}{\partial x_j} \right] \in O(n).$$

Thus Taylor's expansion tells us

$$F = Ox + b,$$

with $O \in O(n), b \in \mathbb{R}^n$.

□

2.4 Show that if X is a vector field of constant length on a Riemannian manifold, then $\nabla_v X$ is always perpendicular to X.

Proof.

$$0 = D_v g(X, X) = 2g(\nabla_v X, X).$$

□

2.5 For any $p \in (M, g)$ and orthonormal basis $e_1, \cdots, e_n$ for $T_p M$, show that there is an orthonormal frame $E_1, \cdots, E_n$ in a neighborhood of p such that $E_i = e_i$ and $(\nabla E)|_p = 0$.

Proof. Fix an orthonormal frame $\overline{E}_i$ near $p \in M$ with $\overline{E}_i(p) = e_i$. If we define $E_i = \alpha_i^j \overline{E}_j$, where $[\alpha_i^j(x)] \in SO(n)$ and $\alpha_i^j(p) = \delta_i^j$, then this will yield the desired frame provided that the $D_{e_k}\alpha_i^j$ are prescribed as

$$0 = D_{e_k}(\alpha_i^j \overline{E}_j) = D_{e_k}\alpha_i^j \overline{E}_j + \alpha_i^j D_{e_k}\overline{E}_j.$$

□

2.7 Let M be a n-dimensional submanifold of $\mathbb{R}^{n+m}$ with the induced metric and assume that we have a local coordinate system given by a parametrization $x^s(u^1, \cdots, u^n), s = 1, \cdots, n+m$. Show that in these coordinates we have:

- $$g_{ij} = \sum_{s=1}^{n+m} \frac{\partial x^s}{\partial u^i}\frac{\partial x^s}{u^j}.$$

- $$\Gamma_{ij,k} = \sum_{s=1}^{n+m} \frac{\partial x^s}{\partial u^k}\frac{\partial^2 x^s}{\partial u^i \partial u^j}.$$

- R_{ijkl} depends only on the first and second partials of x^s.

Proof.
- $$\begin{aligned} g_{ij} &= g\left(dx^s\left(\frac{\partial}{\partial u^i}\right), dx^s\left(\frac{\partial}{\partial u^j}\right)\right) \\ &= g\left(\frac{\partial x^s}{\partial u^i}\frac{\partial}{\partial x^s}, \frac{\partial x^t}{\partial u^j}\frac{\partial}{\partial x^t}\right) \\ &= \frac{\partial x^s}{\partial u^i}\frac{\partial x^t}{\partial u^j}\delta_{st} \\ &= \frac{\partial x^s}{\partial u_i}\frac{\partial x^s}{\partial u_j}. \end{aligned}$$

-

$$\begin{aligned}
\Gamma_{ij,k} &= \Gamma_{ij}^{l} g_{lk} \\
&= \frac{1}{2}(\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}) \\
&= \frac{1}{2}\left[\partial_j(\partial_i x^s \partial_k x^s) + \partial_i(\partial_j x^s \partial_k x^s) - \partial_k(\partial_i x^s \partial_j x^s)\right] \\
&= \frac{1}{2}(\partial_{ij}^2 x^s \partial_k x^s + \partial_k x^s \partial_{ij}^2 x^s) \\
&= \partial_k x^s \partial_{ij}^2 x^s.
\end{aligned}$$

-

$$\begin{aligned}
R_{ijkl} &= g(R(\partial_i, \partial_j)\partial_k, \partial_l) \\
&= g(\nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i} k - \nabla_{[\partial_i,\partial_j]}\partial_k, \partial_l) \\
&= g(\nabla_i(\Gamma_{jk}^{p}\partial_p) - \nabla_j(\Gamma_{ik}^{p}\partial_p), \partial_l) \\
&= \partial_i \Gamma_{jk}^{p} g_{pl} + \Gamma_{jk}^{p}\Gamma_{ip}^{q} g_{ql} - \partial_j \Gamma_{jik}^{p} g_{pl} - \Gamma_{ik}^{p}\Gamma_{jp}^{q} g_{ql},
\end{aligned}$$

while the terms involving third partials of x^s offset:

$$\partial_p x^s \partial_i(\partial_{jk}^2 x^s) - \partial_p x^s \partial_j(\partial_{ik}^2 x^s) = 0.$$

$\square$

2.8 Show that $Hess f = \nabla df$.

Proof.

$$\begin{aligned} Hessf(X,Y) &= g(\nabla_X \nabla f, Y) \\ &= Xg(\nabla f, Y) - g(\nabla f, \nabla_X Y) \\ &= \nabla_X(df(Y)) - df(\nabla_X Y) \\ &= (\nabla_X df)(Y) \\ &= (\nabla df)(X,Y). \end{aligned}$$

□

2.10 Let (M, g) be oriented and define the Riemannian volume form $dvol$ as follows:

$$dvol(e_1, \cdots, e_n) = \det(g(e_i, e_j)) = 1,$$

where $e_1, \cdots, e_n$ is a positively oriented orthonormal basis for T_pM.

- Show that if $v_1, \cdots, v_n$ is positively oriented, then

$$dvol(v_1, \cdots, v_n) = \sqrt{\det(g(v_i, v_j))}.$$

- Show that the volume form is parallel.
- Show that in positively oriented coordinates,

$$dvol = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n.$$

- If X is a vector field, show that

$$L_X dvol = div(X) dvol.$$

- Conclude that the Laplacian has the formula:
$$\triangle u = \frac{1}{\sqrt{\det(g_{ij})}}\partial_k \left(\sqrt{\det(g_{ij})}g^{kl}\partial_l u\right).$$
Given that the coordinates are normal at p we get as in Euclidean space that
$$\triangle f(p) = \sum_{i=1}^{n} \partial_i \partial_i f.$$

Proof. • Let $v_i = \alpha_i^j e_j$, then

$$\begin{aligned} dvol(v_1, \cdots, v_n) &= \sum \alpha_1^{i_1} \cdots \alpha_n^{i_n} \det(e_{i_1}, \cdots, e_{i_n}) \\ &= \sum sgn(i_1, \cdots, i_n)\alpha_1^{i_1} \cdots \alpha_n^{i_n} \\ &= \det \alpha_i^j \\ &= \sqrt{\det \alpha_i^j \cdot \det \alpha_j^i} \\ &= \sqrt{\det(\alpha_i^k \alpha_k^j)} \\ &= \sqrt{\det(g(v_i, v_j)}. \end{aligned}$$

- By Exercise 5, $\exists$ local orthonormal frame (E_i) around p, such that
$$E_i(p) = e_i, \quad \nabla E_i(p) = 0.$$
Then for any $X \in \mathcal{X}(M)$, $X = X^i E_i$, we have
$$\begin{aligned} &(\nabla_X dvol)(E_1, \cdots, E_n) \\ = \ & X(dvol(E_1, \cdots, E_n)) - \sum_d vol(E_1, \cdots, \nabla_X E_i, \cdots, E_n) \\ = \ & 0. \end{aligned}$$

- This is just a direct consequence of the first assertion, i.e.
$$dvol(\partial_1, \cdots, \partial_n) = \sqrt{\det(g_{ij})}$$

implies that

$$dvol = \sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^n.$$

- $$\begin{aligned} & (L_X dvol)(E_1, \cdots, E_n) \\ = \ & L_X(dvol(E_1, \cdots, E_n)) - \sum dvol(E_1, \cdots, L_X E_i, \cdots, E_n) \\ = \ & (divX) dvol(E_1, \cdots, E_n), \end{aligned}$$

  where we use the fact

  $$L_X E_i = [X, E_i] = \nabla_X E_i - \nabla_{E_i} X = -\nabla_{e_i} X = -(\nabla_{e_i} X^i) e_i.$$

- $$\triangle u \cdot dvol = div(\nabla u) dvol = L_{\nabla u} dvol = L_{g^{kl}\partial_l u \partial_k} dvol$$

implies that

$$\begin{aligned} & \triangle u \cdot dvol(\partial_1, \cdots, \partial_n) \\ = \ & (L_{g^{kl}\partial_l u \partial_k} dvol)(\partial_1, \cdots, \partial_n) \\ = \ & g^{kl}\partial_l u (L_{\partial_k} dvol)(\partial_1, \cdots, \partial_n) \\ & + d(g^{kl}\partial_l u)(\partial_m) dvol(\partial_1, \cdots, \partial_k, \cdots, \partial_n) \\ = \ & g^{kl}\partial_l u \partial_k \sqrt{\det(g_{ij})} + \partial_k(g^{kl}\partial_l u)\sqrt{\det(g_{ij})} \\ = \ & \partial_k \left(\sqrt{\det(g_{ij})} g^{kl} \partial_l u \right) \\ = \ & \frac{1}{\sqrt{\det(g_{ij})}} \partial_k \left(\sqrt{\det(g_{ij})} g^{kl} \partial_l u \right) dvol(\partial_1, \cdots, \partial_n), \end{aligned}$$

i.e.

$$\triangle u = \frac{1}{\sqrt{\det(g_{ij})}} \partial_k \left(\sqrt{\det(g_{ij})} g^{kl} \partial_l u \right).$$

In normal coordinates around p,

$$\triangle u = \sum \partial_i \partial_i f.$$

$\square$

2.11 Let (M, g) be a oriented Riemannian manifold with volume form *dvol* as above.

- If f has compact support, then

$$\int_M \triangle f \cdot dvol = 0.$$

- Show that

$$div(f \cdot X) = g(\nabla f, X) + f \cdot divX.$$

- Establish the integration by parts formula for functions with compact support:

$$\int_M f_1 \cdot \triangle f_2 \cdot dvol = -\int_M g(\nabla f_1, \nabla f_2) \cdot dvol.$$

- Conclude that if f is sub- or superharmonic (i.e. $\triangle f \geq 0$ or $\triangle f \leq 0$) then f is constant. This result is known as the weak maximum principle. More generally, one can show that any subharmonic (respectively superharmonic) function that has a global maximum (respectively minimum) must be constant. This result is usually referred to as the strong maximum principle.

Proof.

- $$\begin{aligned}\int_M \triangle f \cdot dvol &= \int_M L_{\nabla f} dvol\\ &= \int_M i_{\nabla f} d(dvol) + d(i_{\nabla f} dvol)\\ &= 0.\end{aligned}$$

- $$\begin{aligned}& div(f \cdot X)\\ =\ & div(f \cdot X) dvol(e_1, \cdots, e_n)\\ =\ & (L_{f \cdot X} dvol)(E_1, \cdots, E_n)\\ =\ & f(L_X dvol)(E_1, \cdots, E_n) + df(E_i) dvol(E_1, \cdots, X, \cdots, E_n)\\ =\ & f(divX) dvol(E_1, \cdots, E_n) + g(\nabla f, E_i) g(X, E_i)\\ =\ & f \cdot divX + g(\nabla f, X).\end{aligned}$$

- $$\begin{aligned}\triangle(f_1 \cdot f_2) &= div(\nabla(f_1 \cdot f_2))\\ &= div(f_1 \cdot \nabla f_2 + f_2 \cdot \nabla f_1)\\ &= f_1 \triangle f_2 + g(\nabla f_1, \nabla f_2) + f_2 \triangle f_1 + g(\nabla f_2, \nabla f_1)\\ &= f_1 \triangle f_2 + 2g(\nabla f_1, \nabla f_2) + f_2 \triangle f_1.\end{aligned}$$

- $$\begin{aligned}\int_M f_1 \cdot \triangle f_2 \cdot dvol &= \int_M f_1 \cdot div(\nabla f_2) \cdot dvol\\ &= \int_M (div(f_1 \cdot \nabla f_2) - g(\nabla f_1, \nabla f_2)) \cdot dvol\\ &= -\int_M g(\nabla f_1, \nabla f_2) \cdot dvol.\end{aligned}$$

- If $\triangle f \geq 0$, then
$$0 = \int_M \triangle f \cdot dvol \geq 0,$$
this implies
$$\triangle f = 0.$$
And hence
$$0 = \int_M f \cdot \triangle f \cdot dvol = -\int_M g(\nabla f, \nabla f) \cdot dvol,$$
$$\nabla f = 0,$$
i.e. f is constant.

For the proof of the strong maximum principle, see P280 of the book.

□

2.13 Let X be a unit vector field on (M, g) such that $\nabla_X X = 0$.

- Show that X is locally the gradient of a distance function iff the orthogonal distribution is integrable.
- Show that X is the gradient of a distance function in a neighborhood of $p \in M$ iff the orthogonal distribution has an integral submanifold through p.
- Find X with the given conditions so that it is not a gradient field.

Proof.
- Let $X, Y_2, \cdots, Y_n$ be orthonormal frame on M, and θ_X be defined as
$$\theta_X(Y) = g(X, Y), \quad \forall\ Y \in \mathcal{X}(M),$$

be the 1-form dual to X. $\Rightarrow$: If X is locally the gradient of a distance function, i.e. $X = \nabla r$ for some $r : U(\subset M) \to \mathbb{R}$. Then

$$\theta_X(Y) = g(X,Y) = g(\nabla r, Y) = dr(Y), \quad \forall\, Y \in \mathcal{X}(M),$$

i.e. $\theta_X = dr$,$d\theta_X = d \circ dr = 0$. Hence

$$\begin{aligned} 0 &= d\theta_X(Y_i, Y_j) \\ &= Y_i(\theta_X(Y_j)) - Y_j(\theta_X(Y_i)) - \theta_X([Y_i, Y_j]) \\ &= -g(X, [Y_i, Y_j]), \end{aligned}$$

i.e.

$$[Y_i, Y_j] = \sum c_{ij}^k Y_k, \text{ for some } c_{ij}^k.$$

$\Leftarrow$: If the distribution $Y = \{Y_2, \cdots, Y_n\}$ is integrable, then

$$g([Y_i, Y_j], X) = 0.$$

We claim that $d\theta_X = 0$ then.

✓

$$d\theta_X(X,X) = X\theta_X(X) - X\theta_X(X) - \theta_X([X,X]) = 0,$$

✓

$$\begin{aligned} & d\theta_X(X, Y_i) = X\theta_X(Y_i) - Y_i\theta_X(X) - \theta_X(X, Y_i) \\ = &\ Xg(X, Y_i) - Yg(X,X) - g(X, [X,Y]) \\ = &\ g(\nabla_X X, Y_i) + g(X, \nabla_X Y_i) \\ &\ -g(\nabla_{Y_i} X, X) - g(X, \nabla_{Y_i} X) \\ &\ -g(X, [X, Y_i]) \\ = &\ g(\nabla_X X, Y) - \frac{1}{2} Y g(X,X) \\ = &\ 0, \end{aligned}$$

✓

$$d\theta_X(Y_i, Y_j) = Y_i\theta_X(Y_j) - Y_j\theta_X(Y_i) - \theta_X([Y_i, Y_j]) = 0.$$

Next, we shall show $X = \nabla r$ for some $r : U(\subset M) \to \mathbb{R}$.

Indeed, in local coordinates, written $X = X^i\partial_i$, we have

$$\begin{aligned} 0 &= d\theta_X(\partial_i, \partial_j) \\ &= \partial_i g(X, \partial_j) - \partial_j g(X, \partial_i) \\ &= \partial_i(X^k g_{kj}) - \partial_j(X^k g_{ki}). \end{aligned}$$

Then a simple mathematical analysis leads to the fact that

$$X^k g_{ki} = \partial_i r, i = 1, 2, \cdots, n,$$

for some $r : U(\subset M) \to \mathbb{R}$. Hence

$$X = X^i\partial_i = g^{ij}\partial_j r\partial_i = \nabla r,$$

as desired.

- This is just a consequence of the first assertion and the Frobenius integrability Theorem for vector fields.
- We consider $S^3 = SU(2)$ with bi-invariant metric, so that

$$g(X_i, X_j) = \delta_{ij},$$

where

$$X_1 = \begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Our claim is then that $\nabla_{X_1} X_1 = 0$ while $[X_2, X_3] = 2X_1$. Thus X_1 is not locally a gradient field.

✓ $[X_2, X_3] = 2X_1$.

Indeed, $[X_i, X_{i+1}] = 2X_i$ (indices are mod 3).

✓ $\nabla_{X_1} X_1 = 0$.

This follows from the Koszul Formula and the Lie bracket just determined.

$\square$

2.14 Given an orthonormal frame $E_1, \cdots, E_n$ on (M, g), define the structure constants c_{ij}^k by $[E_i, E_j] = c_{ij}^k E_k$. Then define the Γs and Rs as

$$\nabla_{E_i} E_j = \Gamma_{ij}^k E_k,$$

$$R(E_i, E_j)E_k = R_{ijk}^l E_l$$

and compute them in terms of cs. Notice that on Lie groups with left-invariant metrics the structure constants can be assumed to be constant. In this case, computations simplify considerably.

Proof. • Γ_{ij}^k is just computed by Koszul Formula.

$$\begin{aligned}
2\Gamma_{ij}^k &= 2g(\nabla_{E_i} E_j, E_k) \\
&= E_i g(E_j, E_k) + E_j g(E_k, E_i) - E_k g(E_i, E_j) \\
&\quad + g(E_k, [E_i, E_j]) + g(E_j, [E_k, E_i]) - g(E_i, [E_j, E_k]) \\
&= c_{ij}^k + c_{ki}^j - c_{jk}^i.
\end{aligned}$$

•

$$\begin{aligned}
R_{ijk}^l &= \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{jk}^s \Gamma_{is}^l - \Gamma_{ik}^s \Gamma_{js}^l \\
&= \cdots.
\end{aligned}$$

□

2.15 There is yet another effective method for computing the connection and curvatures, namely, the Cartan formalism. Let (M, g) be a Riemannian manifold. Given a frame $E_1, \cdots, E_n$, the connection can be written

$$\nabla E_i = \omega_i^j E_j,$$

where ω_i^j are 1-forms. Thus,

$$\nabla_v E_i = \omega_i^j(v) E_j.$$

Suppose now that the frame is orthonormal and let ω^i be the dual coframe, i.e. $\omega^i(E_j) = \delta_j^i$. Show that the connection forms satisfy

$$\omega_i^j = -\omega_j^i,$$

$$d\omega^i = \omega^j \wedge \omega_j^i.$$

These two equations can, conversely, be used to compute the connection forms given the the orthonormal frame. Therefore, if the metric is given by declaring a certain frame to be orthonormal, then this method can be very effective in computing the connection.

If we think of $[\omega_i^j]$ as a matrix, then it represents 1-form with values in the skew-symmetric $n \times n$ matrices, or in other words, with values n the Lie algebra $\mathfrak{so}(n)$ for $O(n)$.

The curvature forms Ω_i^j are 2-forms with values in $\mathfrak{so}(n)$. They are defined as

$$R(\cdot, \cdot) E_i = \Omega_i^j E_j.$$

Show that they satisfy

$$d\omega_i^j = \omega_i^k \wedge \omega_k^j + \Omega_i^j.$$

When reducing to Riemannian metrics on surfaces we obtain for an orthonormal frame E_1, E_2 with coframe ω^1, ω^2

$$d\omega^1 = \omega^2 \wedge \omega_2^1,$$

$$d\omega^2 = -\omega^1 \wedge \omega_2^1,$$

$$d\omega_2^1 = \Omega_2^1,$$

$$\Omega_2^1 = sec \cdot dvol.$$

Proof.

- $$\omega_i^k = \omega_i^j g_{jk} = g(\nabla E_i, E_k) = -g(E_i, \nabla E_k) = -g_{il}\omega_k^l = -\omega_k^i.$$

- $$\begin{aligned} d\omega^i(X,Y) &= X(\omega^i(Y)) - Y(\omega^i(X)) - \omega^i([X,Y]) \\ &= X(g(E_i,Y)) - Yg(E_i,X)) - g(E_i,[X,Y]) \\ &= g(\nabla_X E_i, Y) + g(E_i, \nabla_X Y) \\ &\quad -g(\nabla_Y E_i, X) - g(E_i, \nabla_Y X) \\ &\quad -g(E_i,[X,Y]) \\ &= g(\nabla_X E_i, Y) - g(\nabla_Y E_i, X) \\ &= \omega_i^j(X) g(E_j, Y) - \omega_i^j(Y) g(E_j, X) \\ &= \omega^j(X)\omega_j^i(Y) - \omega^j(Y)\omega_j^i(X) \\ &= (\omega^j \wedge \omega_j^i)(X,Y). \end{aligned}$$

- One may calculate as before, but here we note that both sides of

$$d\omega_i^j = \omega_i^k \wedge \omega_k^j + \Omega_i^j$$

are tensors. We need only to check

$$\begin{aligned}
d\omega_i^j(E_m, E_l) &= E_m(\omega_i^j(E_l)) - E_l(\omega_i^j(E_m)) \\
&= E_m(g(\nabla_{E_l} E_i, E_j)) - E_l(g(\nabla_{E_m} E_i, E_j)) \\
&= g(\nabla_{E_m}\nabla_{E_l} E_i, E_j) + g(\nabla_{E_l} E_i, \nabla_{E_m} E_j) \\
&\quad -g(\nabla_{E_l}\nabla_{E_m} E_i, E_j) - g(\nabla_{E_m} E_i, \nabla_{E_l} E_j) \\
&= g(R(E_m, E_l)E_i, E_j) \\
&\quad +g(\nabla_{E_l} E_i, \nabla_{E_m} E_j) - g(\nabla_{E_m} E_i, \nabla_{E_l} E_j) \\
&= \Omega_i^j(E_m, E_l) + \omega_i^k(E_l)\omega_j^k(E_m) - \omega_i^k(E_m)\omega_j^k(E_l) \\
&= \omega_i^k(E_m)\omega_k^j(E_l) - \omega_i^k(E_l)\omega_k^j(E_m) + \Omega_i^j(E_m, E_l) \\
&= (\omega_i^k \wedge \omega_k^j + \Omega_i^j)(E_m, E_l).
\end{aligned}$$

- In two dimensional case,

$$\Omega_2^1(E_1, E_2) = g(R(E_1, E_2)E_2, E_1) = sec,$$

thus

$$\Omega_2^1 = sec \cdot dvol.$$

□

2.16 Show that a Riemannian manifold with parallel Ricci curvature has constant scalar curvature. In Chapter 3, it will be shown that the converse is not true, and also that a metric with parallel Ricci curvature doesn't have to be Einstein.

Proof. • $dscal = 2divRic$.

We calculate at a fixed point $p \in M$, choose a normal orthonormal frame $E_1, \cdots, E_n$ at p, i.e. $\nabla E_i(p) = 0$. For $\forall X \in \mathcal{X}(M)$,

$$\begin{aligned}
dscal(X) &= X(g(Ric(E_i), E_i)) \\
&= X(g(R(E_i, E_j)E_j, E_i)) \\
&= g((\nabla_X R)(E_i, E_j)E_j, E_i) \\
&= -g((\nabla_{E_i} R)(E_j, X)E_j, E_i) \\
&\quad -g((\nabla_{E_j} R)(X, E_i)E_j, E_i) \\
&= g((\nabla_{E_i} R)(X, E_j)E_j, E_i) \\
&\quad +g((\nabla_{E_j} R)(X, E_i)E_i, E_j) \\
&= 2g((\nabla_{E_i} R)(X, E_j)E_j, E_i) \\
&= 2g(\nabla_{E_i}(Ric(X)) - Ric(\nabla_{E_i} X), E_i) \\
&= 2g((\nabla_{E_i} Ric)X), E_i) \\
&= 2(divRic)(X).
\end{aligned}$$

- $\nabla Ric = 0 \Rightarrow dscal = 0 \Rightarrow scal$ is constant.

□

2.17 Show that if R is the $(1-3)$-curvature tensor and Ric is the $(0, 2)$-Ricci tensor, then

$$(divR)(X, Y, Z) = (\nabla_X Ric)(Y, Z) - (\nabla_Y Ric)(X, Z).$$

Conclude that $divR = 0$ if $\nabla Ric = 0$. Then show that $divR = 0$ iff the $(1, 1)$-Ricci tensor satisfies:

$$(\nabla_X Ric)(Y) = (\nabla_Y Ric)(X), \quad \forall\ X, Y \in \mathcal{X}(M).$$

Proof. • $(divR)(X,Y,Z) = (\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(X,Z)$.

$$\begin{aligned}
& (divR)(X,Y,Z) \\
= \ & g((\nabla_{E_i} R)(X,Y,Z),E_i) \\
= \ & -g((\nabla_X R)(Y,E_i,Z),E_i) - g((\nabla_Y R)(E_i,X,Z),E_i) \\
= \ & -X(g(R(Y,E_i)Z,E_i)) \\
& +g(R(\nabla_X Y,E_i)Z,E_i) + g(R(Y,E_i)\nabla_X Z,E_i) \\
& +\cdots \\
= \ & X(Ric(Y,Z)) - Ric(\nabla_X Y,Z) - Ric(Y,\nabla_X Z) + \cdots \\
= \ & (\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(X,Z).
\end{aligned}$$

• $\nabla Ric = 0 \Rightarrow divR = 0$.

• $divR = 0 \Leftrightarrow [(\nabla_X Ric)(Y) = (\nabla_Y Ric)(X),\ \forall\ X,Y \in \mathcal{X}(M).]$

Just note that

$$\begin{aligned}
& g((\nabla_X Ric)Y,Z) \\
= \ & X(Ric(Y,Z)) - Ric(\nabla_X Y,Z) - Ric(Y,\nabla_X Z) \\
= \ & (\nabla_X Ric)(Y,Z).
\end{aligned}$$

□

2.20 Suppose we have two Riemannian manifolds (M,g_M) and (N,g_N). Then the product has a natural product metric $(M\times N, g_M+g_N)$. Let X be a vector field on M and Y one on N, show that if we regard these as vector fields on $M\times N$, then $\nabla_X Y = 0$. Conclude that $sec(X,Y) = 0$. This means that product metrics always have many curvatures that are zero.

Proof. • $\nabla_X Y = 0$.

This is easily done by Koszul forumla. Indeed, for any $Z \in \mathcal{X}(M), W \in X(N)$,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &\quad - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle \\ &= 0, \end{aligned}$$

$$2g(\nabla_X Y, W) = \cdots = 0.$$

• $sec(X, Y) = 0$.

$$\begin{aligned} sec(X, Y) &= \frac{g(R(X, Y)Y, X)}{|X|^2|Y|^2} \\ &= \frac{g(\nabla_X\nabla_Y Y - \nabla_Y\nabla_X Y - \nabla_{[X,Y]}Y, X)}{|X|^2|Y|^2} = 0. \end{aligned}$$

□

2.24 The Einstein tensor on a Riemannian manifold is defined as

$$G = Ric - \frac{scal}{2} \cdot I.$$

Show that $G = 0$ in dimension 2 and that $divG = 0$ in higher dimensions. This tensor is supposed to measure the mass/engery distribution. The fact that it is divergence free tells us that energy and momentum are conserved. In a vacuum, one therefore imagines that $G = 0$. Show that this happens in dimensions> 2 iff the metric is Ricci flat.

Proof. • In dimension 2,

$$sec(e_1, e_2) = R_{1221} = \langle Ric(e_1), e_1\rangle = \langle Ric(e_2), e_2\rangle,$$

$$scal = \langle Ric(e_1), e_1 \rangle + \langle Ric(e_2), e_2 \rangle = 2R_{1221},$$

where e_1, e_2 orthonormal at a given point p of M.

Thus

$$G(e_1) = Ric(e_1) - \frac{scal}{2} e_1 = R_{1221} e_1 - R_{1221} e_1 = 0,$$

$$G(e_2) = \cdots = 0.$$

- In dimensions ≥ 3,

$$divG = divRic - div\left(\frac{scal}{2} \cdot I\right) = \frac{dscal}{2} - \frac{dscal}{2} = 0.$$

Indeed,

$$\begin{aligned} div(scal \cdot I)(e_i) &= \sum_j \left\langle \nabla_{e_j}(scal \cdot I), e_j \right\rangle (e_i) \\ &= \sum_j \left\langle \left(\nabla_{e_j}(scal \cdot I)\right) e_i, e_j \right\rangle \\ &= \sum_j \left\langle \nabla_{e_j}(scal \cdot e_i), e_j \right\rangle \\ &= \sum_j \left\langle (\nabla_{e_j} scal) e_i, e_j) \right\rangle \\ &= \nabla_{e_i} scal = dscal(e_i). \end{aligned}$$

Note that we calculate at a normal neighborhood at a given point.

- $G = 0 \Leftrightarrow Ric = 0$ if $n \geq 3$.

Indeed, if $G = 0$, then $Ric = \frac{scal}{2} \cdot I$, taking contractions imply that

$$scal = \frac{n}{2} scal,$$

thus if $n \geq 3$, $scal = 0$, $Ric = \frac{scal}{2} \cdot I = 0$.

□

2.25 This exercise will give you a way of finding the curvature tensor from the sectional s curvatures. Using the Bianchi identity show that

$$\begin{aligned} -6R(X,Y,Z,W) &= \frac{\partial^2}{\partial s\partial t}|_{s=t=0}\left\{R(X+sZ,Y+tW,Y+tW,X+sZ)\right. \\ &\quad \left.-R(X+sW,Y+tZ,Y+tZ,X+sW)\right\}. \qquad (2.1) \end{aligned}$$

Proof. Since

$$\begin{aligned} & R(X+sZ,Y+tW,Y+tW,X+sZ) \\ = \ & st\left\{R(Z,W,Y,X)+R(Z,Y,W,X)+R(X,W,Y,Z)+R(X,Y,W,Z)\right\} \\ & +\cdots \\ = \ & -2stR(X,Y,W,Z)+2stR(Z,Y,W,X)+\cdots, \end{aligned}$$

we have

$$\begin{aligned} & \frac{\partial^2}{\partial s\partial t}|_{s=t=0}R(X+sZ,Y+tW,Y+tW,X+sZ) \\ = \ & -2R(X,Y,W,Z)+2R(Z,Y,W,X). \end{aligned}$$

Thus

$$\begin{aligned} \text{R.H.S. of Eq. (2.1)} &= -2R(X,Y,W,Z)+2R(Z,Y,W,X) \\ &\quad +2R(X,Y,Z,W)-2R(W,Y,Z,X) \\ &= -4R(X,Y,Z,W) \\ &\quad +2\left(R(Z,Y,W,X)+R(Y,W,Z,X)\right) \\ &= -6R(X,Y,Z,W) \\ &= \text{L.H.S. of Eq. (2.1)}. \end{aligned}$$

□

3 Examples

3.4 The Heisenberg group with its Lie algebra is

$$G = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\},$$

$$\mathfrak{g} = \left\{ \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$$

A basis for the Lie algebra is:

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Show that the only nonzero bracket are

  $$[X, Y] = -[Y, X] = Z.$$

  Now introduce a left-invariant metric on G such that X, Y, Z form an orthonormal frame.

- Show that the Ricci tensor has both negative and positive eigenvalues.
- Show that the scalar curvature is constant.
- Show that the Ricci tensor is not parallel.

Proof. Due to the fact

$$A \in \mathfrak{g} \Leftrightarrow e^{tA} \in G,$$

we have the elements of $\mathfrak{g}$ are upper triangle matrices.

- Since $XY = Z, YX = 0; XZ = 0, ZX = 0; YZ = 0, ZY = 0$; we deduce that

  $$[X, Y] = -[Y, X] = Z,$$

  while other brackets being zero.

- Applying Koszul formula, we have

  $$\nabla_X Y = Z; \nabla_X Z = \nabla_Z X = -Y; \nabla_Y Z = \nabla_Z Y = X;$$

  while other connections being zero.

  Hence

  $$Ric(X) = R(X, Y)Y + R(X, Z)Z = -2X + X = -X;$$

  $$Ric(Y) = R(Y, X)X + R(Y, Z)Z = -Y + Y = 0;$$

  $$Ric(Z) = Ric(Z, X)X + R(Z, Y)Y = Z + 0 = Z.$$

  Thus the eigenvalues of Ric are $-1, 0, 1$.

- $scal = Ric(X, X) + Ric(Y, Y) + Ric(Z, Z) = -1 + 0 + 1 = 0$. Aha, the Heisenberg group is scalar flat.

- Since

  $$\begin{aligned} (\nabla_X Ric)(Y, Z) &= \nabla_X(Ric(Y, Z)) - Ric(\nabla_X Y, Z) - Ric(Y, \nabla_X Z) \\ &= 0 - 1 - 0 = -1, \end{aligned}$$

  we gather that $\nabla Ric \neq 0$, the Ricci tensor is not parallel.

$\square$

3.5 Let $\tilde{g} = e^{2\psi} g$ be a metric conformally equivalent to g. Show that

- $$\tilde{\nabla}_X Y = \nabla_X Y + ((D_X\psi)Y + (D_Y\psi)X - g(X,Y)\nabla\psi).$$

- If X, Y are orthonormal with respect to g, then

$$\begin{aligned} e^{2\psi}\widetilde{sec}(X,Y) &= sec(X,Y) - Hess\psi(X,X) - Hess\psi(Y,Y) \\ &\quad -|\nabla\psi|^2 + (D_X\psi)^2 + (D_Y\psi)^2. \end{aligned}$$

Proof. • Again, we invoke the Koszul formula,

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_X Y, Z) &= X\tilde{g}(Y,Z) + \cdots - \tilde{g}(X,[Y,Z]) + \cdots \\ &= X(e^{2\psi}g(Y,Z)) + \cdots - e^{2\psi}g(X,[Y,Z]) + \cdots \\ &= 2(\nabla_X\psi)\tilde{g}(Y,Z) + 2(\nabla_Y\psi)\tilde{g}(Y,Z) + \cdots \\ &\quad -e^{2\psi}g(X,[Y,Z]) + \cdots \\ &= 2(\nabla_X\psi)\tilde{\tilde{g}}(Y,Z) + 2(\nabla_Y\psi)\tilde{g}(Z,X) - 2(\nabla_Z\psi)\tilde{g}(X,Y) \\ &\quad +\tilde{g}(\nabla_X Y, Z) \\ &= 2\tilde{g}((\nabla_X\psi)Y + (\nabla_Y\psi)Z - g(X,Y)\nabla\psi + \nabla_X Y, Z), \end{aligned}$$

where in the last inequality, we use the following fact:

$$\nabla_Z\psi = d\psi(Z) = g(\nabla\psi, Z),$$

and

$$\begin{aligned} (\nabla_Z\psi)\tilde{g}(X,Y) &= g(\nabla\psi, Z)\tilde{g}(X,Y) \\ &= \tilde{g}(\nabla\psi, Z)g(X,Y) = \tilde{g}(g(X,Y)\nabla\psi, Z). \end{aligned}$$

- Note that if X, Y are orthonormal w.r.t. g, then

$$\widetilde{sec}(X,Y)=\frac{\tilde{g}(\tilde{R}(X,Y)Y,X)}{\tilde{g}(X,X)\tilde{g}(Y,Y)}=e^{-2\psi}g(\tilde{R}(X,Y)Y,X),$$

i.e.

$$\begin{aligned} e^{2\psi}\widetilde{sec}(X,Y) &= g(\tilde{R}(X,Y)Y,X) \\ &= g(\tilde{\nabla}_X\tilde{\nabla}_YY-\tilde{\nabla}_Y\tilde{\nabla}_XY-\tilde{\nabla}_{[X,Y]}Y,X). \quad (3.1) \end{aligned}$$

We just need to calculate each term on the R.H.S. of Eq. (3.1).

★ Calculation of $g(\tilde{\nabla}_X\tilde{\nabla}_YY,X)$.

$$\tilde{\nabla}_YY=\nabla_YY+2(\nabla_Y\psi)Y-\nabla\psi;$$

$$\begin{aligned} \tilde{\nabla}_X\tilde{\nabla}_YY &= \nabla_X\left(\nabla_YY+2(\nabla_Y\psi)Y-\nabla\psi\right) \\ &\quad +(\nabla_X\psi)\left(\nabla_YY+2(\nabla_Y\psi)Y-\nabla\psi\right) \\ &\quad \left(\nabla_{\nabla_YY+2(\nabla_Y\psi)Y-\nabla\psi}\psi\right)X \\ &\quad -g(X,\nabla_YY+2(\nabla_Y\psi)Y-\nabla\psi)\psi; \end{aligned}$$

$$\begin{aligned} & g(\tilde{\nabla}_X\tilde{\nabla}_YY,X) \\ = & \ g(\nabla_X\nabla_YY,X)+2(\nabla_Y\psi)g(\nabla_XY,X)-g(\nabla_X\nabla\psi,X) \\ & +(\nabla_X\psi)g(\nabla_YY,X)-|\nabla_X\psi|^2 \\ & +g(\nabla_YY,\nabla\psi)+2|\nabla_Y\psi|^2-|\nabla\psi|^2 \\ & -g(X,\nabla_YY)(\nabla_X\psi)+|\nabla_X\psi|^2 \\ = & \ g(\nabla_X\nabla_YY,X)+2(\nabla_Y\psi)g(\nabla_XY,X)-Hess\psi(X,X) \\ & +\left(\nabla_Y\nabla_Y\psi-Hess\psi(Y,Y)\right)-|\nabla\psi|^2+2|\nabla_Y\psi|^2. \quad (3.2) \end{aligned}$$

★ Calculation of $g(\tilde{\nabla}_Y\tilde{\nabla}_X Y, X)$.

$$\tilde{\nabla}_X Y = \nabla_X Y + (\nabla_X\psi)Y + (\nabla_Y\psi)X;$$

$$\begin{aligned}
\tilde{\nabla}_Y\tilde{\nabla}_X Y &= \nabla_Y\left(\nabla_X Y + (\nabla_X\psi)Y + (\nabla_Y\psi)X\right) \\
&\quad +(\nabla_Y\psi)\left(\nabla_X Y + (\nabla_X\psi)Y + (\nabla_Y\psi)X\right) \\
&\quad +\left(\nabla_{\nabla_X Y+(\nabla_X\psi)Y+(\nabla_Y\psi)X}\psi\right)Y \\
&\quad -g(Y, \nabla_X Y + (\nabla_X\psi)Y + (\nabla_Y\psi)X)\nabla\psi;
\end{aligned}$$

$$\begin{aligned}
& g(\tilde{\nabla}_Y\tilde{\nabla}_X Y, X) \\
=\ & (g(\nabla_Y\nabla_X Y, X) + (\nabla_X\psi)g(\nabla_Y Y, X) \\
& +\nabla_Y\nabla_Y\psi + (\nabla_X\psi)g(\nabla_Y X, X)) \\
& +(\nabla_Y\psi)g(\nabla_X Y, X) + |\nabla_Y\psi|^2 \\
& -g(Y, \nabla_X Y)(\nabla_X\psi) - |\nabla_X\psi|^2 \\
=\ & g(\nabla_Y\nabla_X Y, X) + (\nabla_X\psi)g(\nabla_Y Y, X) + \nabla_Y\nabla_Y\psi \\
& +(\nabla_Y\psi)g(\nabla_X Y, X) + |\nabla_Y\psi|^2 \\
& -|\nabla_X\psi|^2. \qquad (3.3) \\
& \left(g(Y, \nabla_X Y) = \frac{1}{2}X|Y|^2 = 0; g(\nabla_Y X, X) = \cdots = 0\right)
\end{aligned}$$

★ Calculation of $g(\tilde{\nabla}_{[X,Y]}Y, X)$.

$$\begin{aligned}
\tilde{\nabla}_{[X,Y]}Y &= \nabla_{[X,Y]}Y + (\nabla_{[X,Y]}\psi)Y \\
&\quad +(\nabla_Y\psi)[X, Y] - g([X, Y], Y)\nabla\psi;
\end{aligned}$$

$$
\begin{aligned}
& g(\tilde{\nabla}_{[X,Y]}Y, X) \\
= \; & g(\nabla_{[X,Y]}Y, X) \\
& +(\nabla_Y \psi) g(\nabla_X Y, X) - (\nabla_Y \psi) g(\nabla_Y X, X) \\
& -(\nabla_X \psi) g(\nabla_X Y, Y) + (\nabla_X \psi) g(\nabla_Y X, Y) \\
= \; & g(\nabla_{[X,Y]}Y, X) + (\nabla_Y \psi) g(\nabla_X Y, X) + (\nabla_X \psi) g(\nabla_Y X, Y)
\end{aligned}
\tag{3.4}
$$

Combining Eqs. (3.2),(3.3), (3.4), and substituting into Eq. (3.1), we gather that

$$
\begin{aligned}
e^{2\psi}\widetilde{sec}(X,Y) = \; & sec(X,Y) - Hess\psi(X,X) - Hess\psi(Y,Y) \\
& -|\nabla\psi|^2 + (D_X\psi)^2 + (D_Y\psi)^2,
\end{aligned}
$$

as required.

□

4 Hypersurfaces

4.4 Let (M, g) be a closed Riemannian manifold, and suppose that there is a Riemannian embedding into $\mathbb{R}^{n+1}$. Show that there must be a point $p \in M$ where the curvature operator $\mathfrak{R} : \wedge^2 T_pM \to \wedge T_pM$ is positive.

Proof. This is geometrically obvious, but the analytical proof is as follows.

Let

$$
\begin{aligned}
f : \quad \mathbb{R}^n \quad & \to \quad \mathbb{R} \\
x \mapsto \quad & |x|^2.
\end{aligned}
$$

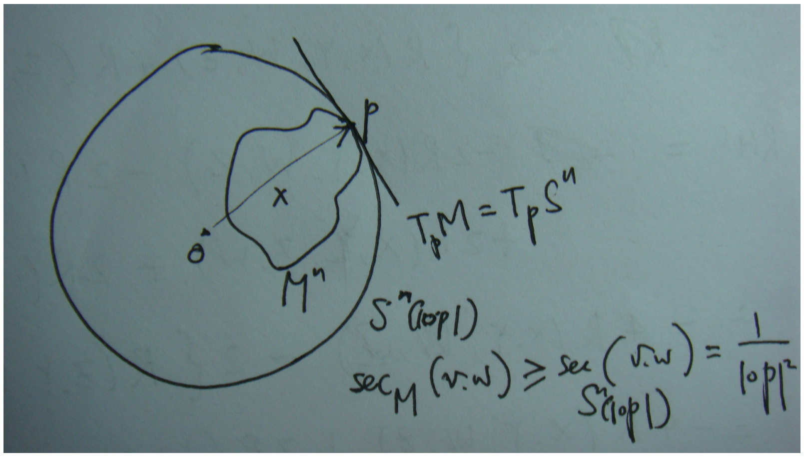

Figure 2: Curvature comparison between M and S^n

Then since M is closed, $f|_M$ attains its maximum at $p \in M$.

Claim $x \perp T_pM$.

Indeed, by Exercise 5.9,

$$0 = \langle \nabla f, v \rangle = \langle Df, v \rangle = 2 \langle x, v \rangle, \ \forall\ v \in T_pM.$$

Here and thereafter, we use the notation:

★ ∇: the connection on M,

★ D: the connection on $\mathbb{R}^n$.

Now, choose an orthonormal basis $\{e_i\}$ of T_pM such that

$$D_{e_i} \frac{x}{|x|} = S(e_i) = \lambda_i e_i,$$

and let E_i be the orthonormal extension on M of e_i around p.

Differentiating

$$\langle \nabla, E_i \rangle = \langle Df, E_i \rangle = 2 \langle x, E_i \rangle$$

in the direction E_i, we obtain

$$\langle \nabla_{E_i} \nabla f, E_i \rangle + \langle \nabla f, \nabla_{E_i} E_i \rangle = 2 \langle D_{E_i} X, E_i \rangle + 2 \langle X, D_{E_i} E_i \rangle . \quad (4.1)$$

While at $p \in M$,

$$\text{L.H.S. of Eq. (4.1)} \quad = \quad Hessf(E_i, E_i) \leq 0;$$

$$\begin{aligned} & \text{R.H.S. of Eq. (4.1)} \\ = \quad & 2 \left\langle D_{E_i} \left(\frac{x}{|x|} |x| \right), E_i \right\rangle - 2 \left\langle x, \nabla_{E_i}^{S^n(|x|)} E_i - D_{E_i} E_i \right\rangle \\ = \quad & 2|x| \left\langle D_{e_i} \frac{x}{|x|}, E_i \right\rangle - 2|0p| \left\langle \frac{x}{|x|}, II^{S^n(|x|)}(E_i, E_i) \right\rangle \\ = \quad & 2|x| \langle S(E_i), E_i \rangle + 2|x| \left\langle S^{S^n(|op|)}(E_i), E_i \right\rangle \\ & (\text{ Here we use the notation as in Exercise 5.8 }) \\ = \quad & 2|x|\lambda_i + 2|x| \cdot \frac{1}{|x|} \\ = \quad & 2 (|x|\lambda_i + 1) . \end{aligned}$$

Thus we gather that

$$\lambda_i \leq -\frac{1}{|op|},$$

$$\begin{aligned} sec(e_i, e_j) \quad &= \quad \langle S(e_i), e_i \rangle \langle S(e_j), e_j \rangle - |\langle S(E_i), E_j \rangle|^2 \\ &= \quad \lambda_i \lambda_j \\ &\geq \quad \frac{1}{|op|^2} > 0. \end{aligned}$$

$\square$

4.5 Suppose (M, g) is immersed as a hypersurface in $\mathbb{R}^{n+1}$, with shape operator S.

- Using the Codazzi-Mainardi equations, show that

$$divS = d(trS).$$

- Show that if $S = f(x) \cdot I$ for some function f, then f must be a constant and the hypersurface must have constant curvature.
- Show that $S = \lambda \cdot Ric$ iff the metric has constant curvature.

Proof.

- We calculate in a normal neighborhood as:

$$\begin{aligned}(divS)(E_i) &= \sum_j \left\langle \nabla_{E_j} S, E_j \right\rangle (E_i) = \sum_j \left\langle (\nabla_{E_j} S)(E_i), E_j \right\rangle \\ &= \sum_j \langle (\nabla_{E_i} S)(E_j), E_j \rangle = \sum_j \langle \nabla_{E_i}(S(E_j)), E_j \rangle \\ &= \sum_j \nabla_{E_i} \langle S(E_j), E_j \rangle = \nabla_{E_i} trS = d(trS)(E_i).\end{aligned}$$

- If $S = f(x) \cdot I$, then

$$df = divS = d(trS) = d(nf).$$

Thus $(n-1)df = 0$. Since $n > 1$ (we consider this case), $df = 0$, $f \equiv const$. And $S = const \cdot I$,

$$\begin{aligned}sec(E_i, E_j) &= \langle R(E_i, E_j)E_j, E_i \rangle \\ &= \langle S(E_i, E_i \rangle \langle S(E_j), E_j \rangle - \langle S(E_i), E_j \rangle \langle S(E_j), E_i \rangle \\ &= const^2.\end{aligned}$$

- $\Rightarrow$ If $S = \lambda \cdot Ric$, by Codazzi-Mainardi equations and Exercise 2.17, we have $divR = 0$, thus

  $\Leftarrow$ If (M, g) has constant curvature, then by Exercise 2.17 again,

  $$(\nabla_X Ric)(Y) = (\nabla_Y Ric)(X). \tag{4.2}$$

  We now have another identity:

  $$\begin{aligned} \langle R(X,Y)Z, W\rangle &= \lambda^2 \langle Ric(X), W\rangle \langle Ric(Y), Z\rangle \\ &\quad -\lambda^2 \langle Ric(X), Z\rangle \langle Ric(Y), W\rangle, \end{aligned} \tag{4.3}$$

  for some constant $\lambda \in \mathbb{R}$.

  A tedious calculation may verify, using the polarization identity like Exercise 2.25.

  Now, the fundamental theorem of Hypersurface theory tells us (by Eqs. (4.2), (4.3)) that $\lambda \cdot Ric = S'$ for some shape operator of M, but M is already immersed in $\mathbb{R}^{n+1}$, we have

  $$\lambda \cdot Ric = S' = S.$$

$\square$

5 Geodesics and Distance

5.2 A Riemannian manifold is said to be homogeneous if the isometry group acts transitively. Show that homogeneous manifolds are geodesically complete.

Proof. For any $p \in M$, $v \in T_pM$ with $|v| = 1$, let γ be the geodesic with data (p, v). Denote by $\mathcal{T}*$ the maximal existence time for γ. Then we have the

Claim $\mathcal{T}^* = \infty$.

Indeed, if $\mathcal{T}^* < \infty$, let

- $\varepsilon > 0$ be such that

$$\exp_p : \overline{B(0, 2\varepsilon)} \subset T_pM \to \overline{B_{2\varepsilon}(0)} \subset M$$

  is a diffeomorphism,

- $F \in Iso(M, g)$ with

$$F(p) = \gamma(\mathcal{T}^* - \varepsilon),\ w \triangleq (dF^{-1})_{\gamma(\mathcal{T}^* - \varepsilon)} \dot{\gamma}(\mathcal{T}^* - \varepsilon).$$

Now since

$$|w| = |\dot{\gamma}(\mathcal{T}^* - \varepsilon)| = |\dot{\gamma}(0)| = |v| = 1,$$

there is a geodesic $\tilde{\gamma} : [0, 2\varepsilon] \to M$ with data (p, w). Hence $F(\tilde{\gamma})$ is a geodesic with data $(\gamma(\mathcal{T}^* - \varepsilon), \dot{\gamma}(\mathcal{T}^* - \varepsilon))$. Indeed,

$$0 = \big(DF(\nabla_{\dot{\tilde{\gamma}}} \dot{\tilde{\gamma}}) = \nabla_{(F(\tilde{\gamma}))'} F(\tilde{\gamma})\big)$$

While uniqueness of ode tells us that

$$\sigma = \begin{cases} \gamma, & \text{on } [0, \mathcal{T}^* - \varepsilon], \\ F(\tilde{\gamma}), & \text{on } [\mathcal{T}^* - \varepsilon, \mathcal{T}^* + \varepsilon], \end{cases}$$

is a geodesic with data (p, v). This contradicts the definition of $\mathcal{T}^*$.

Finally the proof is complete if we invoke the classical Hopf-Rinow theorem and notice the homogeneity of geodesics. $\square$

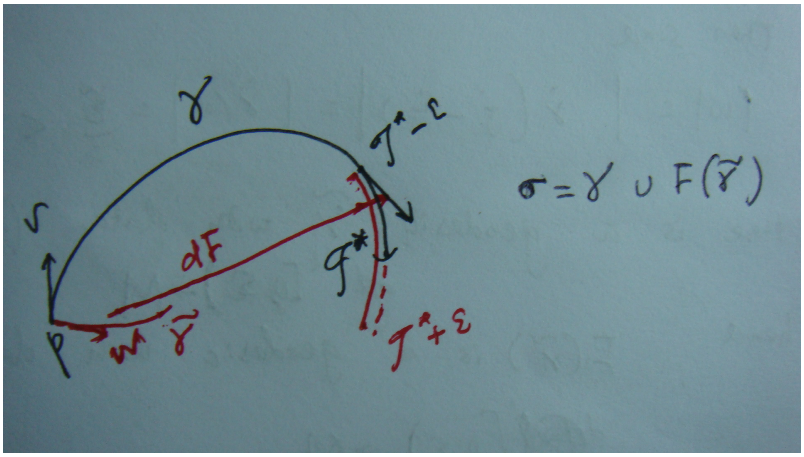

Figure 3: the composed geodesic

5.8 Let $N \subset (M, g)$ be a submanifold. Let ∇^N denote the connection on N that comes from the metric induced by g. Define the second fundamental form of N in M by

$$II(X, Y) = \nabla_X^N Y - \nabla_X Y.$$

- Show that $II(X, Y)$ is symmetric and hence tensorial in X and Y.
- Show that $II(X, Y)$ is always normal to N.
- Show that $II = 0$ on N iff N is totally geodesic.
- If $\mathbb{R}^N$ is the curvature form of N, then

$$\begin{aligned} g(R(X, Y)Z, W) &= g(R^N(X, Y)Z, W) \\ &-g(II(Y, Z), II(X, W)) + g(II(X, Z), II(Y, W)). \end{aligned}$$

Proof. • Due to the fact

$$\begin{aligned} II(X,Y) &= \nabla^N_X Y - \nabla_X Y = \big([X,Y] + \nabla^N_Y X\big) - ([X,Y] + \nabla_Y X) \\ &= \nabla^N_Y X - \nabla_Y X = II(Y,X), \end{aligned}$$

we see that II is symmetric. And by definition of the connection, II is tensorial in X, thus tensorial in Y as

$$\begin{aligned} & II(X, fY_1 + gY_2) = II(fY_1 + gY_2, X) \\ = \; & fII(Y_1, X) + gII(Y_2, X) = fII(X, Y_1) + gII(X, Y_2). \end{aligned}$$

- Indeed, Koszul formula tells us that $\nabla^N_X Y = (\nabla_X Y)^\top$, where $\top$ is the projection from TM to TN, thus

$$II(X,Y) = (\nabla_X Y)^\top - \nabla_X Y = (\nabla_X Y)^\perp,$$

which is normal to N.

- Recall that N is totally geodesic in M iff any geodesic in N is a geodesic in M. Now we prove the assertion.

$\Rightarrow$ If $II = 0$ and γ is a geodesic in N, then $\nabla^N_{\dot\gamma}\dot\gamma = 0$, thus $\nabla_{\dot\gamma}\dot\gamma = 0$, γ is a geodesic.

$\Leftarrow$ By the formula

$$II(X,Y) = \frac{1}{2}\left[II(X+Y, X+Y) - II(X,X) - II(Y,Y)\right],$$

we need only to show that $II(X,X) = 0$, $\forall\ X \in \mathcal{X}(M)$. But II is tensorial, we are redirected to prove that

$$II(v,v) = 0,\ \forall\ v \in T_pN, \forall\ p \in N.$$

This is obviously true. In fact, for any $p \in N$, $v \in T_pN$, let γ be the geodesic in N with initial data (p, v), then $\nabla^N_{\dot\gamma}\dot\gamma = 0$, and by hypothesis, $\nabla_{\dot\gamma}\dot\gamma = 0$, $II(v, v) = 0$.

$\square$

5.9 Let $f : (M, g) \to \mathbb{R}$ be a smooth function on a Riemannian manifold.

- If $\gamma : (a, b) \to M$ is a geodesic, compute the first and second derivative of $f \circ \gamma$.
- Use this to show that at a local maximum (or minimum) for f the gradient is zero and the Hessian nonpositive (or nonnegative).
- Show that f has everywhere nonnegative Hessian iff $f \circ \gamma$ is convex for all geodesics γ in (M, g).

Proof.

- We omit the subscript for simplicity.

$$\frac{d}{ds}(f \circ \gamma) = Df(\dot\gamma) = df(\dot\gamma) = D_{\dot\gamma} f = g(\nabla f, \dot\gamma),$$

$$\begin{aligned} \frac{d^2}{ds^2}(f \circ \gamma) &= \frac{d}{ds}\left(\frac{d}{ds}(f \circ \gamma)\right) = D_{\dot\gamma}(D_{\dot\gamma} f) = D_{\dot\gamma}(Df(\dot\gamma)) \\ &= (D_{\dot\gamma}(Df))(\dot\gamma) + Df(D_{\dot\gamma}\dot\gamma) \\ &= (D(Df))(\dot\gamma, \dot\gamma) = D^2 f(\dot\gamma, \dot\gamma). \end{aligned}$$

- We consider the case when f attains its local minimum at $p \in M$. Then for $\forall\ v \in T_pM$, let γ be the geodesic with initial data (p, v), we have

$$0 = \frac{d}{ds}(f \circ \gamma) = g(\nabla f, \dot\gamma) = g(\nabla f, v),$$

and

$$0 \geq \frac{d^2}{ds^2}(f \circ \gamma) = Hessf(\dot{\gamma}, \dot{\gamma}) = Hessf(v, v),$$

at p. Hence the conclusion.

- We just take the following equiv.:

$$\begin{aligned} & Hessf(v,v) \geq 0, \ \forall\ v \in T_pM \\ \Leftrightarrow\ & \frac{d^2}{ds^2}(f \circ \gamma) \geq 0, \ \forall\ \gamma \text{ geodesic} \\ \Leftrightarrow\ & f \circ \gamma \text{ is convex}, \forall\ \gamma \text{ geodesic.} \end{aligned}$$

□

5.12 Compute the cut locus on a sphere and real projective space with constant curvature metrics.

Proof. We consider the case $(S^n, can_{\mathbb{R}^n}|_{S^n})$ with curvature 1. For any $p \in S^n$, $cut(p) = \{-p\}$. While for $\mathbb{R}P^n$ (What's the meaning of the problem? Is it mean that $\mathbb{R}P^n$ is given a metric so that it is of constant curvature or ...), $cut([p])$ =the equator.

□

6 Sectional Curvature Comparison I

6.1 Show that in even dimension the sphere and real projective space are the only closed manifolds with constant positive curvature.

Proof. If M is of even dimension, closed (compact and without boundary), and with positive curvature,

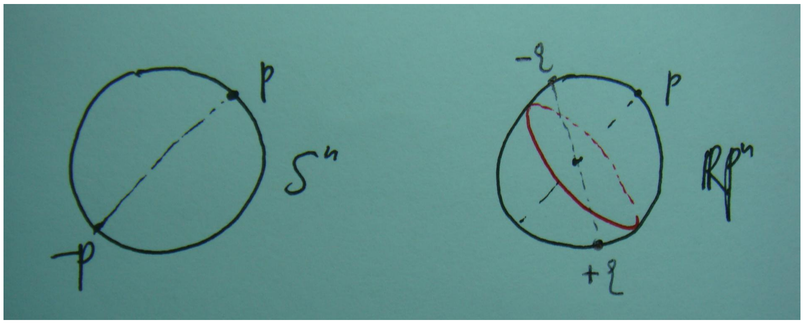

Figure 4: Cut locus of S^n and $\mathbb{R}P^n$

- in case M is orientable, then Synge theorem tells us that M is simple connected, thus M are spheres;
- in case M is non-orientable, then the orientable double covering of M are spheres, thus M are real projective spaces.

□

6.5 Let $\gamma : [0,1] \to M$ be a geodesic. Show that $exp_{\gamma(0)}$ has a critical point at $t\dot{\gamma}(0)$ iff there is a Jacobi field J along γ such that $J(0) = 0$, $\dot{J}(0) \neq 0$, and $J(t) = 0$.

Proof. We assume w.l.o.g. that $t = 1$. First note that

$$\begin{aligned} & exp_p \text{ has a critical point at } \dot{\gamma}(0) \\ \Leftrightarrow \quad & \exists\, 0 \neq w \in T_{\dot{\gamma}(0)} T_{\gamma(0)} M, s.t.\ (dexp_{\gamma(0)})_{\dot{\gamma}(0)}(w) = 0. \end{aligned}$$

$\Rightarrow$ Let $J(t) = (dexp_{\gamma(0)})_{t\dot{\gamma}(0)}(tw)$, $t \in [0,1]$, then J is the Jacobi field we are chasing.

$\Leftarrow$ If we have a Jacobi field $J(t)$ as in the problem, then

$$(dexp_{\gamma(0)})_{\dot{\gamma}(0)}(\dot{J}(0)) = J(1) = 0,$$

with $0 \neq \dot{J}(0) \in T_{\dot{\gamma}(0)}(T_{\gamma(0)}M)$. $\square$

6.8 Let γ be geodesic and X be a Killing field in a Riemannian manifold. Show that the restriction of X to γ is a Jacobi field.

Proof. Recall that

$$X \text{ Killing field } \Leftrightarrow L_X g = 0.$$

Now let $\{e_1 = \dot{\gamma}, e_2, \cdots, e_n\}$ be the parallel orthonormal vector fields along γ, then

$$\begin{aligned} 0 &= (L_X g)(\dot{\gamma}, e_i) \\ &= D_X(g(\dot{\gamma}, e_i)) - g(L_X\dot{\gamma}, e_i) - g(\dot{\gamma}, L_X e_i) \\ &= -g(\nabla_X\dot{\gamma}, e_i) + g(\nabla_{\dot{\gamma}}X, e_i) - g(\dot{\gamma}, \nabla_X e_i) + g(\dot{\gamma}, \nabla_{e_i}X) \\ &= g(\nabla_{\dot{\gamma}}X, e_i) + g(\dot{\gamma}, \nabla_{e_i}X). \qquad (6.1) \end{aligned}$$

In particular,

$$g(\nabla_{\dot{\gamma}}X, \dot{\gamma}) = 0. \qquad (6.2)$$

Now differentiating Eq. (6.1) w.r.t. $\dot{\gamma}$, we find that

$$\begin{aligned}
0 &= D_{\dot{\gamma}}\left(g(\nabla_{\dot{\gamma}}X, e_i)\right) + D_{\dot{\gamma}}\left(g(\dot{\gamma}, \nabla_{e_i}X)\right) \\
&= g(\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}X, e_i) + g(\dot{\gamma}, \nabla_{\dot{\gamma}}\nabla_{e_i}X) \\
&= g(\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}X, e_i) + g(\dot{\gamma}, \nabla_{\dot{\gamma}}\nabla_{e_i}X - \nabla_{e_i}\nabla_{\dot{\gamma}}X - \nabla_{[\dot{\gamma}, e_i]}X) \quad (6.3) \\
&= g(\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}X, e_i) + g(R(\dot{\gamma}, e_i)X, \dot{\gamma}) \\
&= g(\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}X, e_i) + g(R(X, \dot{\gamma})\dot{\gamma}, e_i) \\
&= g(\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}X + R(X, \dot{\gamma})\dot{\gamma}, e_i).
\end{aligned} \tag{6.4}$$

Hence

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}X + R(X, \dot{\gamma})\dot{\gamma} = 0,$$

i.e. X is a Jacobi field along γ.

Note that in Eq. (6.3), we have used the following fact:

$$\begin{aligned}
& g(\dot{\gamma}, \nabla_{e_i}\nabla_{\dot{\gamma}}X + \nabla_{[\dot{\gamma}, e_i]}X) \\
=\ & D_{e_i}\left\{g(\dot{\gamma}, \nabla_{\dot{\gamma}}X)\right\} - g(\nabla_{e_i}\dot{\gamma}, \nabla_{\dot{\gamma}}X) - g(\dot{\gamma}, \nabla_{\nabla_{e_i}\dot{\gamma}}X) \\
=\ & -\left(g(\nabla_{e_i}\dot{\gamma}, \nabla_{\dot{\gamma}}X) + g(\dot{\gamma}, \nabla_{\nabla_{e_i}\dot{\gamma}}X)\right) \text{ (by (6.2))} \\
=\ & 0.\text{(skew-symmetric property of Killing fields)}
\end{aligned}$$

□

6.21 (The Index Form) Below we shall use the second variation formula to prove several results established in Chapter 5. If V, W are vector fields along a geodesic $\gamma : [0,1] \to (M, g)$, then the index form is the symmetric bilinear form

$$I_0^1(V, W) = I(V, W) = \int_0^1 \left(g(\dot{V}, \dot{W}) - g(R(V, \dot{\gamma})\dot{\gamma}, W)\right) dt.$$

In case the vector fields come from a proper variation of γ this is equal to the second variation of energy. Assume below that $\gamma : [0,1] \to (M, g)$ locally minimize the energy functional. This implies that $I(V,V) \geq 0$ for all proper variations.

- If $I(V,V) = 0$ for a proper variation, then V is a Jacobi field.
- Let V and J are variational fields along γ such that $V(0) = J(0)$ and $V(1) = J(1)$. If J is a Jacobi field show that
$$I(V,J) = I(J,J).$$
- (The Index Lemma) Assume in addition that there are no Jacobi fields along γ that vanish at both end points. If V and J are both as above. Show that $I(V,V) \geq I(J,J)$ with equality holding only if $V = J$ on $[0,1]$.
- Assume that there is a nontrivial Jacobi field J that vanishes at 0 and 1. Show that $\gamma : [0, 1+\varepsilon] \to M$ is not locally minimizing for $\varepsilon > 0$.

Proof. Note that the vector fields we consider are all smooth.

- For any proper variational filed W (i.e. $W(0) = 0 = W(1)$),
$$\begin{aligned} 0 &\leq I(V+\varepsilon W, V+\varepsilon W) \\ &= I(V,V) + 2\varepsilon I(V,W) + \varepsilon^2 I(W,W) \\ &= \varepsilon\left[2I(V,W) + \varepsilon I(W,W)\right]. \end{aligned}$$
Letting $\varepsilon \to 0^+, 0^-$, we get $I(V,W) = 0$. Thus
$$0 = I(V,W) = -\int_0^1 g\left(\ddot{V} + R(V,\dot{\gamma})\dot{\gamma}, W\right) dt,$$

and hence $\ddot{V} + R(V, \dot{\gamma})\dot{\gamma} = 0$, V is a Jacobi field.

- This follows from direct computation as

$$\begin{aligned} I(V-J, J) &= \int_0^1 \Big(g(\dot{V} - \dot{J}, \dot{J}) - g(R(V-J, \dot{\gamma})\dot{\gamma}, J) \Big)\, dt \\ &= -\int_0^1 \Big(g(V-J, \ddot{J}) + g(V-J, R(J, \dot{\gamma})\dot{\gamma}) \Big)\, dt \\ &\quad \text{(Here we use the boundary conditions...)} \\ &= -\int_0^1 g\left(V-J, \ddot{J} + R(J, \dot{\gamma})\dot{\gamma} \right) dt = 0. \end{aligned}$$

- If $V \neq J$, then $V - J$ is a proper variational field.

  Claim $0 < I(V-J, V-J) = I(V, V) - I(J, J)$.

  Indeed, if $I(V-J, V-J)$, then the first assertion tells us that $V-J$ is a nontrivial, proper Jacobi field, contradicting the hypotheses.

- See the figure attached and one may compute as

$$\begin{aligned} 0 &= I_0^1(J, J) = I_0^{1+\varepsilon}(J, J) \\ &= I_0^{1-\varepsilon}(J, J) + I_{1-\varepsilon}^{1+\varepsilon}(J, J) \\ &> I_0^{1-\varepsilon}(J, J) + I_{1-\varepsilon}^{1+\varepsilon}(K, K) \\ &\quad \text{(Here we use the Index Lemma)} \\ &= I(V, V). \end{aligned}$$

$\square$

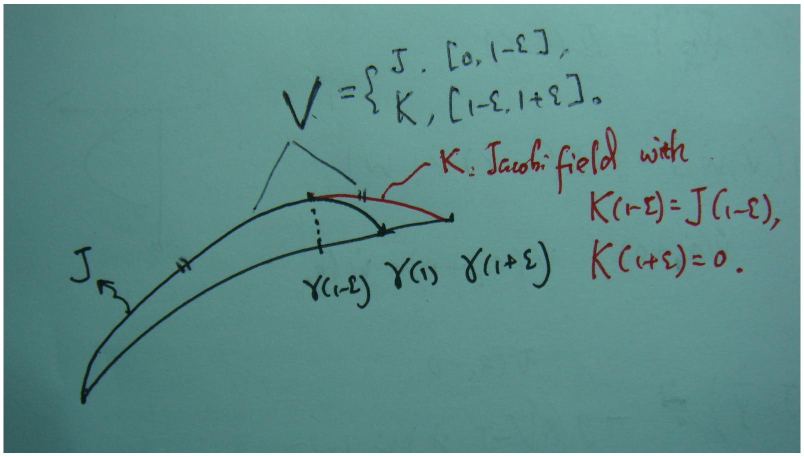

Figure 5: the composed variational field

Concluding Remarks

Thanks to the inspiring and fantastic lectures of Professor Zhu, from whom the author learnt a lot.

But due to

- the author's limited knowledge,
- the fast process of this tedious work,

errors or even blunders may occur. So any comments, whether

- critical, i.e. the reference answer is wrong(?), not accurate(?), misprints(?), e.t.c.
- constructive, i.e. there are beautiful proofs, e.t.c.

is welcome.

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