# Analytical solution of average path length for Apollonian networks

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With the help of recursion relations derived from the self-similar structure, we obtain the solution of average path length, $\bar{d}_t$, for Apollonian networks. In contrast to the well-known numerical result $\bar{d}_t \propto (\ln N_t)^{3/4}$ [J. S. Andrade, Jr. *et al.*, Phys. Rev. Lett. **94**, 018702 (2005)], our rigorous solution shows that the average path length grows logarithmically as $\bar{d}_t \propto \ln N_t$ in the infinite limit of network size $N_t$. The extensive numerical calculations completely agree with our closed-form solution.

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One of the most important properties of complex networks is average path length (APL), which is the mean length of the shortest paths between all pairs of vertices (nodes) [1]. Most real networks have been shown to be small-world or ultra-small-world networks [2–5], that is, their APL $d$ behaves as a logarithmic or double logarithmic scaling with the network size $N$, $d \sim \ln N$ [6] or $d \sim \ln \ln N$ [7]. It has been established that APL is relevant in many fields regarding real-life networks. In the design or interpretation of routes in architectural design, signal integrity in communication networks, the propagation of diseases or beliefs in social networks or of technology in industrial networks, APL is a natural network statistic to compute and interpret. It is strongly believed that many processes such as routing, searching, and spreading become more efficient when APL is smaller. So far, much attention has been paid to the question of APL [8–13].

Recently, on the basis of the well-known Apollonian packing [14], Andrade *et al.* introduced Apollonian networks [15] which were also proposed by Doye and Massen in Ref. [16] simultaneously. Apollonian networks belong to a deterministic growing type of networks, which have drawn much attention from the scientific communities and have turned out to be a useful tool [17–31]. Many topological properties of Apollonian networks such as degree distribution, clustering coefficient, and correlations have been determined analytically [15,16], and the effects of the Apollonian networks on several dynamical models have been intensively studied, including Ising model and a magnetic model [15,32–34]. Despite the importance and usefulness of the quantity APL, there is no analytical calculations for the APL of Apollonian networks.

In this paper, we derive a formula for the average path length characterizing the Apollonian networks. The analytic method is based on the recursive construction and self-similar structure of Apollonian networks. Our rigorous result shows that APL grows logarithmically with the number of nodes. The obtained analytical solution modifies the previous numerical result in [15], where the authors claimed that the APL of Apollonian networks scales sublogarithmically with network size. Our analytical technique could provide a paradigm for computing the APL of deterministic networks.

The Apollonian network, denoted as $A_t$ ($t \geq 0$) after $t$ generations, is constructed as follows [15]: For $t=0$, $A_t$ is a triangle. For $t \geq 1$, $A_t$ is obtained from $A_{t-1}$. For each of the existing triangles created at step $t-1$, we add a new vertex and join it to all the vertices of this triangle. Alternatively, Apollonian network can be created in another method [35,36]. Given the generation $t$, $A_{t+1}$ may be obtained by joining at four edge nodes three copies of $A_t$, see Fig. 1(a). According to the latter construction algorithm, we can easily compute that the total number of vertices of $A_t$ is $N_t = (3^t + 5)/2$.

Apollonian network presents the typical characteristics of real-life networks in nature and society [6,37]. It has a power-law degree distribution with exponent $\gamma = 1 + \ln 3/\ln 2$ [15,16], which belongs to the interval between 2 and 3. For any individual vertex with degree $k$, its clustering coefficient $C(k)$ is also approximately inversely proportional to its degree $k$ as $C(k) = (6/k) - [2/(k-1)]$. The mean value $C$ of clustering coefficients of all vertices is very large, which asymptotically reaches a constant value of 0.8284. Moreover, the network is small world, and its diameter, defined as the longest shortest path length between all pairs of vertices, increases logarithmically with the number of vertices [38]. In fact, Apollonian network can be expanded to general cases [16,34,38] associated with other self-similar

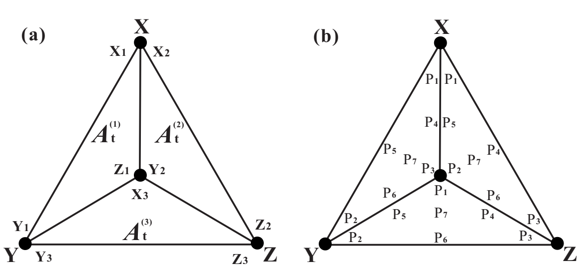


FIG. 1. (a) Second construction method of the Apollonian network. $A_{t+1}$, is obtained by joining three copies of $A_t$ denoted as $A_t^{(\varphi)}$ ($\varphi = 1, 2, 3$), which are connected to one another at the edge nodes (i.e., $X$, $Y$, $Z$, and $O$). Note that the central node is represented as $O$ (marker not visible). (b) Illustration of the recursive definition of node classification. From the node classification of $A_t^{(1)}$, $A_t^{(2)}$, and $A_t^{(3)}$, we can derive recursively the classification of nodes in network $A_{t+1}$.

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packings [39]. Analogously, we can construct the stochastic version of these networks [34,40–42]. The main topological properties of the Apollonian-type networks are controlled by the dimension of packings.

After introducing the Apollonian networks, we now investigate analytically the average path length. We represent all the shortest path lengths of network $A_t$ as a matrix in which the entry $g_{uv}$ is the shortest path length from node $u$ to $v$. A measure of the typical separation between two nodes in $A_t$ is given by the average path length $\bar{d}_t$ defined as the mean of shortest path lengths over all couples of nodes, i.e., $\bar{d}_t = \frac{D_t}{N_t(N_t-1)/2}$, where $D_t = \sum_{u,v\in A_t, u\neq v} g_{uv}$ denotes the sum of the shortest path length between two nodes over all pairs.

By the second construction, it is obvious that the Apollonian network has a self-similar structure. Thus, the total distance $D_{t+1}$ satisfies the recursion relation

$$D_{t+1} = 3D_t + \Theta_t - 3, \tag{1}$$

where $\Theta_t$ is the sum over all shortest path length whose end points are not in the same $A_t$ branch. The solution of Eq. (1) is

$$D_t = 3^{t-1}D_1 + \sum_{m=1}^{t-1} (3^{t-m-1}\Theta_m) - 3^{t-1}. \tag{2}$$

Thus, all that is left to obtain $D_t$ is to compute $\Theta_m$.

The paths that contribute to $\Theta_t$ must all go through at least one of the four edge nodes ($X$, $Y$, $Z$, and $O$) at which the different $A_t$ branches ($A_t^{(1)}$, $A_t^{(2)}$, $A_t^{(3)}$) are connected. The analytical expression for $\Theta_t$, named the crossing path length, can be derived as below.

Denote $\Theta_t^{\alpha,\beta}$ as the sum of all shortest paths with end points in $A_t^{(\alpha)}$ and $A_t^{(\beta)}$. Note that $\Theta_t^{\alpha,\beta}$ rules out the paths with end point at the edge of $A_t^{(\alpha)}$ and $A_t^{(\beta)}$. For example, each path contributed to $\Theta_t^{1,2}$ should not end at node $O$ or $X$. Then the total sum $\Theta_t$ is given by $\Theta_t = \Theta_t^{1,2} + \Theta_t^{1,3} + \Theta_t^{2,3}$. By symmetry, $\Theta_t^{1,2} = \Theta_t^{1,3} = \Theta_t^{2,3}$, so that $\Theta_t = 3\Theta_t^{1,2}$, where $\Theta_t^{1,2}$ is given by the sum

$$\Theta_t^{1,2} = \sum_{u\in A_t^{(1)}, v\in A_t^{(2)}, u,v\neq X \cup u,v\neq O} g_{uv}. \tag{3}$$

To calculate the crossing path length $\Theta_t^{1,2}$, we classify nodes in network $A_{t+1}$ into seven different parts according to their shortest path lengths to each of the three vertices (i.e., $X$, $Y$, $Z$) of the peripheral triangle $\Delta XYZ$. Notice that vertices $X$, $Y$, and $Z$ themselves are not classified into any of the seven parts represented as $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$, and $P_7$, respectively. The classification of nodes is shown in Fig. 1(b). For any interior node $v$, we denote the shortest path lengths from $v$ to $X$, $Y$, $Z$ as $a$, $b$, and $c$, respectively. By construction, $a$, $b$, $c$ can differ by at most 1 since vertices $X$, $Y$, and $Z$ are adjacent. Then the classification function class($v$) of a node $v$ is defined to be

$$\text{class}(v) = \begin{cases} P_1 & \text{for } a < b = c, \\ P_2 & \text{for } b < a = c, \\ P_3 & \text{for } c < a = b, \\ P_4 & \text{for } a = c < b, \\ P_5 & \text{for } a = b < c, \\ P_6 & \text{for } b = c < a, \\ P_7 & \text{for } a = b = c. \end{cases} \tag{4}$$

It should be mentioned that the definition of node classification is recursive. For instance, class $P_1$ and $P_4$ in $A_t^{(1)}$ belongs to class $P_1$ in $A_{t+1}$, class $P_2$ and $P_6$ in $A_t^{(1)}$ belongs to class $P_2$ in $A_{t+1}$, class $P_3$ in $A_t^{(1)}$ belongs to class $P_7$ in $A_{t+1}$, and class $P_5$ and $P_7$ in $A_t^{(1)}$ belongs to class $P_5$ in $A_{t+1}$. Since the three vertices $X$, $Y$, and $Z$ are symmetrical, in the Apollonian network we have the following equivalent relations from the viewpoint of class cardinality: classes $P_1$, $P_2$, and $P_3$ are equivalent to one another, and it is the same with classes $P_4$, $P_5$, and $P_6$. We denote the number of nodes in network $A_t$ that belong to class $P_1$ as $N_{t,P_1}$, the number of nodes in class $P_2$ as $N_{t,P_2}$, and so on. By symmetry, we have $N_{t,P_1} = N_{t,P_2} = N_{t,P_3}$ and $N_{t,P_4} = N_{t,P_5} = N_{t,P_6}$. Therefore, in the following computation we will only consider $N_{t,P_1}$, $N_{t,P_4}$, and $N_{t,P_7}$. It is easy to conclude that

$$\begin{aligned} N_t &= N_{t,P_1} + N_{t,P_2} + N_{t,P_3} + N_{t,P_4} + N_{t,P_5} + N_{t,P_6} + N_{t,P_7} + 3 \\ &= 3N_{t,P_1} + 3N_{t,P_4} + N_{t,P_7} + 3. \end{aligned} \tag{5}$$

Considering the self-similar structure of the Apollonian network, we can easily know that at time $t+1$, the quantities $N_{t+1,P_1}$, $N_{t+1,P_4}$, and $N_{t+1,P_7}$ evolve according to the following recursive equations:

$$\begin{aligned} N_{t+1,P_1} &= 2N_{t,P_1} + 2N_{t,P_4}, \\ N_{t+1,P_4} &= N_{t,P_4} + N_{t,P_7}, \\ N_{t+1,P_7} &= 3N_{t,P_1} + 1, \end{aligned} \tag{6}$$

where we have used the equivalence relations $N_{t,P_1} = N_{t,P_2} = N_{t,P_3}$ and $N_{t,P_4} = N_{t,P_5} = N_{t,P_6}$.

For a node $v$ in network $A_{t+1}$, we are also interested in the smallest value of the shortest path length from $v$ to any of the three peripheral vertices $X$, $Y$, and $Z$. We denote the shortest distance as $f_v$, which can be defined to be $f_v = \min(a,b,c)$. Let $d_{t,P_1}$ denote the sum of $f_v$ of all nodes belonging to class $P_1$ in network $A_t$. Analogously, we can also define the quantities $d_{t,P_2}, d_{t,P_3}, \ldots, d_{t,P_7}$. Again by symmetry, we have $d_{t,P_1} = d_{t,P_2} = d_{t,P_3}$, $d_{t,P_4} = d_{t,P_5} = d_{t,P_6}$, and $d_{t,P_1}$, $d_{t,P_4}$, $d_{t,P_7}$ can be written recursively as follows:

$$\begin{aligned} d_{t+1,P_1} &= 2d_{t,P_1} + 2d_{t,P_4}, \\ d_{t+1,P_4} &= d_{t,P_4} + d_{t,P_7}, \\ d_{t+1,P_7} &= 3(d_{t,P_1} + N_{t,P_1}) + 1. \end{aligned} \tag{7}$$

Next we begin to determine the value of the crossing path length $\Theta_t^{1,2}$. Using the definition of $f_v$, $\Theta_t^{1,2}$ can be rewritten as

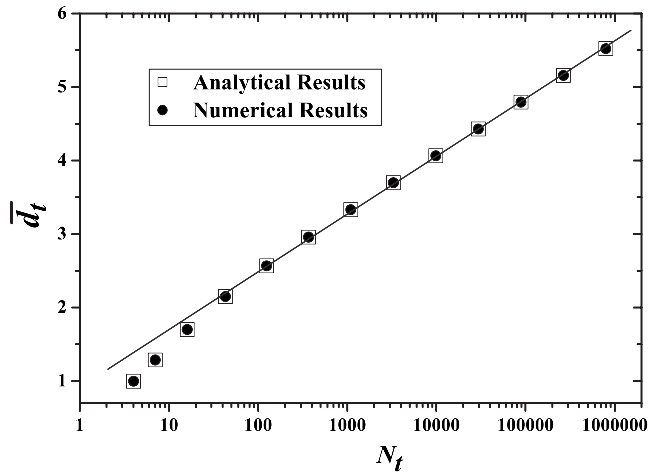


FIG. 2. Average path length $\bar{d}_t$ versus network order $N_t$ on a semilogarithmic scale. The solid line is a guide to the eye, which clearly shows that the APL scales logarithmically with the network size.

$$\Theta_t^{1,2} = \sum_{u\in A_t^{(1)},\, v\in A_t^{(2)},\, u,v\neq X\cup u,\, v\neq O} (f_u + f_v + \delta_{uv}), \tag{8}$$

where $\delta_{uv}$ can be 0 or 1 depending on whether the two paths corresponding to $f_u$ and $f_v$ may meet at the same vertex ($X$ or $O$). If the two paths meet at the same vertex, $\delta_{uv}=0$, $\delta_{uv}=1$ otherwise. According to the above classification of nodes, $\Theta_t^{1,2}$ can be expressed as

$$\Theta_t^{1,2} = \sum_{i=1}^{7}\sum_{j=1}^{7} (N_{t-1,P_j} d_{t-1,P_i} + N_{t-1,P_i} d_{t-1,P_j} + N_{t-1,P_i} N_{t-1,P_j} \delta_{P_iP_j}), \tag{9}$$

where $P_i$ and $P_j$ are node classes of $A_{t-1}^{(1)}$ and $A_{t-1}^{(2)}$, respectively. The quantity $\delta_{P_iP_j}$ may equal to either 0 or 1, which are related to two kinds of paths: paths from nodes in $P_i$ to the three peripheral vertices of $A_{t-1}^{(1)}$, and paths from nodes in $P_j$ to the three peripheral vertices of $A_{t-1}^{(2)}$. If these two sorts of paths may meet at the same peripheral vertex, $\delta_{P_iP_j}=0$; if they do not meet, $\delta_{P_iP_j}=1$. Combining previous equations and results, we obtain the final expression for $\Theta_t^{1,2}$,

$$\begin{aligned}\Theta_t^{1,2} = \frac{1}{2178}\exp(-i\pi t)\Bigg[& 2^t(-62-16\, i\sqrt{2}) + 11\, i2^{t/2}(10\, i + 7\sqrt{2})\exp\left(\frac{i\pi t}{2}\right) - 11\, i2^{t/2}(-10\, i + 7\sqrt{2})\exp\left(\frac{3\, i\pi t}{2}\right) \\ & + i2^{1+t}(31\, i + 8\sqrt{2})\exp(2\, i\pi t) + \exp(i\pi t)(-484 + 11 \times 3^{2+t} + 9^{3+t} + 22\, t \times 9^{1+t})\Bigg].\end{aligned} \tag{10}$$

Inserting Eq. (10) into Eq. (2) and using the initial condition $D_1=6$, we have

$$\begin{aligned} D_t = \frac{1}{330(-3+i\sqrt{2})(-3\, i+\sqrt{2})}[& 6655\, i + 155(-i)^t 2^{1/2+t/2} - 155\, i^t\, 2^{1/2+t/2} + 5i(-i)^t 2^{4+t/2} + 5\, ii^t\, 2^{4+t/2} \\ & + 31\, i(-1)^t 2^{2+t} + 133\, i\, 3^{3+t} + 40\, i\, 3^{2+2t} + 55\, it\, 3^{1+t} + 55\, it\, 3^{1+2t}]. \end{aligned} \tag{11}$$

Then, the analytic expression for average path length can be obtained as

$$\begin{aligned} \bar{d}_t = \frac{4}{1815(15+8\times 3^t + 9^t)}[& 6655 + 5(-i)^t 2^{4+t/2} + 5\, i^t 2^{4+t/2} - 155\, i(-i)^t 2^{1/2+t/2} + 155\, ii^t\, 2^{1/2+t/2} + 31(-1)^t 2^{2+t} \\ & + 133 \times 3^{3+t} + 40 \times 9^{1+t} + 55\, t \times 3^{1+t}(1+3^t)], \end{aligned} \tag{12}$$

which can be simplified according to $t$ as follows:

$$\bar{d}_t = \begin{cases} \dfrac{4[6655 + 5\times 2^{5+t/2} + 31\times 2^{2+t} + 133\times 3^{3+t} + 40\times 9^{1+t} + 55\times 3^{1+t}(1+3^t)t]}{1815(15+8\times 3^t+9^t)}, \\ \dfrac{4[6655 - 155\times 2^{3/2+t/2} - 31\times 2^{2+t} + 133\times 3^{3+t} + 40\times 9^{1+t} + 55\times 3^{1+t}(1+3^t)t]}{1815(15+8\times 3^t+9^t)}, \\ \dfrac{4[6655 - 5\times 2^{5+t/2} + 31\times 2^{2+t} + 133\times 3^{3+t} + 40\times 9^{1+t} + 55\times 3^{1+t}(1+3^t)t]}{1815(15+8\times 3^t+9^t)}, \\ \dfrac{4[6655 + 155\times 2^{3/2+t/2} - 31\times 2^{2+t} + 133\times 3^{3+t} + 40\times 9^{1+t} + 55\times 3^{1+t}(1+3^t)t]}{1815(15+8\times 3^t+9^t)}, \end{cases} \tag{13}$$

for $t \equiv 0,1,2,3 (\text{mod } 4)$ are given consecutively. In the infinite network size (i.e., large $t$), $\bar{d}_t \simeq \frac{4}{11}t \sim \ln N_t$. Thus, the APL grows logarithmically with increasing size of the network. We have checked our analytic result against numerical calculations for different network size up to $t=13$ which corresponds to $N_{13}=797\,164$. In all the cases we obtain a complete agreement between our theoretical formula and the results of numerical investigation, see Fig. 2.

We finally make a comment on the result presented in [15]. Using a combination of numerical experiments and heuristic arguments, Ref. [15] advances claims about the interesting quantity of average path length. The authors of [15] claimed that $\bar{d}_t \propto (\ln N_t)^{3/4}$. They thus concluded that the Apollonian network belongs to a new class of networks which interpolates between *small* ($\bar{d}_t \propto \ln N_t$) and *ultrasmall* ($\bar{d}_t \propto \ln \ln N_t$) networks [7]. While the combinatorial arguments and simulation results are sound, the behavior of APL as $t$ grows is quite difficult to estimate numerically. Just as it is difficult to verify with direct computation that the harmonic series diverges, the experimental results in [15] underestimate the asymptotic behavior of APL. In contrast to previous claims, our obtained precise value for APL shows that the conclusion of [15] seems questionable. The self-similar structure of Apollonian network allows us to compute precisely the quantity APL, which is difficult to apprehend through simulation.

In conclusion, in this paper we have derived analytically the solution for the average path length of Apollonian networks which has been attracting much research interest. We found that in the infinite network size limit Apollonian networks are small world, the APL scales logarithmically with network size. However, some authors have proved that conventional random scale-free networks with degree exponent $\gamma < 3$ show ultra-small-world property [7–9]. In the future, it will be interesting to perform further studies to reveal this dissimilarity between Apollonian networks and those stochastic scale-free networks, which have different scaling for APL. Finally, we believe that the analytical calculation here can guide and shed light on related studies for deterministic network models.

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