Chebyshev Collocation Spectral Methods for Coupled Radiation and Conduction in a Concentric Spherical Participating Medium

The Chebyshev collocation spectral method for coupled radiative and conductive heat transfer in concentric spherical participating medium is introduced and formulated. The angular dependence of the problem is discretized by conventional discrete ordinates method, and the space dependence is expressed by Chebyshev polynomial and discretized by collocation spectral method. Due to the exponential convergence of the spectral methods, very high accuracy can be obtained even using a small resolution (i.e., number of collocation points) for present problems. Comparisons between the solutions from Chebyshev collocation spectral–discrete ordinates method (SP-DOM) with available numerical or exact solutions in references indicate that the SP-DOM for the combination of radiation and conduction in concentric spherical participating medium is accurate and efficient. [DOI: 10.1115/1.3090617]

Keywords: radiation, conduction, combined heat transfer, participating medium, discrete ordinates method, Chebyshev collocation spectral methods

1 Introduction

Coupled radiative and conductive heat transfer in participating media exists in many engineering applications. Various approximate or numerical methods have been developed for this problem. The formulations, methods, and some results for combined radiation and conduction with participating media can be found in Ref. [1]. More recent review on methods for coupled radiative and conductive heat transfer can be found in Ref. [2]. The integrodifferential radiative transfer equation (RTE) with or without participating media can be solved by many numerical methods [1,3], say, zonal method, Monte Carlo method, discrete ordinates method (DOM), finite volume method (FVM), finite element method (FEM), etc. After the determination of spatial distributions of radiative intensities, or further the radiative energy sources, the diffusion equation for energy conservation can be solved by using finite difference method (FDM) or finite element method.

As for coupled radiative and conductive heat transfer special in spherical or in concentric spherical media, Viskanta and Crosbie [4] and Viskanta and Merriam [5] gave the rigorous formulations using exact radiative transfer methods, and the resulting equations were numerically solved by successive approximation. Stewart and Thomas [6] used the spherical harmonic method together with a sphere-to-plane transformation technique to solve this kind of problem, while Jia et al. [7] used Galerkin method, and Sghaier et al. [8] and Trabelsi et al. [9] developed a discrete ordinates method associated with the finite Legendre transform. Recently, Aoued-Diala et al. [10] used the finite Chebyshev transform (FCT) to treat the angular derivative term of the discretized one-dimensional RTE, and finally the resulting equations were solved simultaneously using boundary value problem with finite difference (BVPFD). Kim et al. [11] used the combined finite volume and discrete ordinates method to investigate radiative heat transfer between two concentric spheres.

In the community of computational mechanics or numerical simulations, spectral methods can provide exponential convergence (in other words, spectral accuracy) [12] and have been widely used to solve Navier–Stokes equations in computational fluid dynamics (CFD) [13,14]. Maxwell equations in electrodynamics [15], magnetohydrodynamics (MHD) equations in magnetofluids mechanics [16,17], and so on. Early in 1992, Zenouzi and Yener [18,19] used the Galerkin method to solve the radiative part of a radiation and natural convection combined problem. Later Kuo et al. [20,21] made a numerical comparison for spectral methods and finite volume method to solve the radiation and natural convection combined problem, and they concluded that the spectral methods are more accurate. Ganz et al. [22] used spectral methods for radiative heat transfer in a reacting flow. De Oliveira et al. [23] made a combination of spectral method and Laplace transform to solve radiative heat transfer in isotropic scattering media. Last year, the collocation spectral method for radiative part in stellar modeling was carried out [24]. Very recently, in Modest and Singh’s work [25], the spherical harmonics method ($P_N$ method so called, from mathematical view, the $P_N$ method belongs to the category of spectral methods) was further developed to reduce the number of first-order partial differential equations. Other works regarding radiative transfer may be Refs. [26–30]. However, all these five works are not really correlated with “heat transfer” by thermal radiation but with quantum physics [26], electromagnetic wave [27,28], beam wave [29] propagation and scattering, and neutron transport [30]. Besides this fact, the final resultant algebraic equations in Refs. [26–29] need to be solved in coefficient space; in other words, spectral equations. Very recently, we have successfully developed the Chebyshev collocation method for one-dimensional radiative heat transfer even with anisotropic scattering media [31] and for one-dimensional radiative heat transfer in graded index medium [32]. As an expansion of our work [31,32], we developed the Chebyshev collocation method for coupled radiation and conduction in concentric spherical participating medium.

In the following of this paper, the physical model and governing equation will be presented in Sec. 2. The Chebyshev collocation methods for both RTE and diffusion equation for concentric...
where $q_r(r)$ and $\kappa$ are the radiative heat flux at position $r$ and the thermal conductivity of the medium, respectively.

The divergence of the radiative flux vector in Eq. (3) can be expressed as

$$\frac{d^2 I(r)}{dr^2} + \frac{2}{r} \frac{dI(r)}{dr} = \frac{2}{r} \frac{dq_r(r)}{dr}$$

with boundary conditions

$$T(R_1) = T_1$$
$$T(R_2) = T_2$$

Substituting Eq. (5) into Eq. (3), one obtains

$$\frac{d^2 I(r)}{dr^2} + \frac{2}{r} \frac{dI(r)}{dr} = 4\pi K_r \left[ I_r(r) - \frac{G(r)}{4\pi} \right]$$

$G(r)$ in Eqs. (5) and (6) is the incident radiative energy at position $r$, and $q_r(r)$ and $G(r)$ come from RTE (1).

Similar to Ref. [10], employing the reference absolute temperature, $T_{ref} = T_1$, and the other dimensionless variables, namely, dimensionless temperature $\theta = T/T_1$, dimensionless radius $r' = r/R_2$, dimensionless radiation intensity $\psi = \pi n^2\sigma K_r T_1^4$, dimensionless radiation flux $q'_r = q_r n^2\sigma K_r T_1^4$, conduction-radiation parameter $N_{cr} = K_r \beta / 4n^2\sigma K_r T_1^4$, single scattering albedo $\omega = K_e / \beta$, and optical thickness $\tau_e = \beta R_2$, the set of dimensionless coupled equations governing the combined radiation and conduction problems in spherical medium can be expressed as

$$\frac{\mu}{\tau_e(r'^2)} \frac{\partial}{\partial r'} \left[ (r'^2) \psi \right] + \frac{1}{\tau_e(r'^2)} \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \psi \right] + \psi = (1 - \omega) \theta_d(r') + \frac{\omega}{2} \int_{-1}^{1} \psi(r', \mu') d\mu'$$

with the boundary conditions

$$\psi(R_1', \mu) = \epsilon_1 \theta_1 + 2(1 - \epsilon_1) \int_{-1}^{0} \psi(R_1', \mu') |\mu'| d\mu', \quad \mu > 0$$

and

$$\psi(R_2', \mu) = \epsilon_2 \theta_2 + 2(1 - \epsilon_2) \int_{0}^{1} \psi(R_2', \mu') |\mu'| d\mu', \quad \mu < 0$$

with the boundary conditions

$$\theta(R_1') = 1$$
$$\theta(R_2') = \theta_2$$

3 Discretization of Governing Equations and Boundary Conditions Import

3.1 Formulations of Radiative Transfer Equation. For the sake of simplicity, the superscript "*" on the radius will be canceled hereafter. The discretization of dimensionless RTE (7) includes two aspects, i.e., space and angle. In our work, the DOM is used to treat the angular dependence of the RTE, and $M$ points, $\{\mu^1, \mu^2, \ldots, \mu^M\}$, in the angular domain, $[-1, 1]$, together with the
corresponding weights \( w_m, m=1,2,\ldots,M \), are selected. The integral term on the right hand side of Eq. (7) is replaced by numerical quadrature. After that, Eq. (7) becomes
\[
\frac{\mu^m}{\tau^2} \frac{\partial}{\partial r} (r^2 \psi^m) + \frac{1}{\tau^2 r} \frac{\partial}{\partial \mu} \{ [(1-\mu^2)\psi^m] \}_{\mu=\mu^m} + \psi^m
\]
\[
= (1-\omega) \theta^m(r) + \frac{\alpha}{2} \sum_{m=1}^{M} w_m \psi^m
\]  
(11)
and the corresponding boundary conditions become
\[
\psi^m(R_1) = e_1 \theta_1^m + 2(1-e_1) \sum_{m=1}^{M} w_m \psi^m (R_1) |_{\mu^m} > 0, \quad \mu^m < 0
\]
\[
\psi^m(R_2) = e_2 \theta_2^m + 2(1-e_2) \sum_{m=1}^{M} w_m \psi^m (R_2) |_{\mu^m} > 0, \quad \mu^m > 0
\]  
(12a)
\[
\psi^m = e_1 \theta_1^m + 2(1-e_1) \sum_{m=1}^{M} w_m \psi^m (R_1) |_{\mu^m} > 0, \quad \mu^m < 0
\]  
(12b)
For each direction \( \mu = \mu^m \) in Eq. (11), the angular derivative term is discretized by central difference scheme [3], namely,
\[
\left\{ \frac{\partial}{\partial \mu} [1-\mu^2] \psi^m \right\}_{\mu=\mu^m} = \frac{\alpha^{m+1/2} \psi^{m+1/2} - \alpha^{m-1/2} \psi^{m-1/2}}{w^m}
\]  
(13)
where \( \psi^{m+1/2} \) and \( \psi^{m-1/2} \) are the angular intensities in the directions \( m+1/2 \) and \( m-1/2 \), and the central difference scheme is adopted to correlate them to the unknown \( \psi^m \), i.e., \( \psi^m = \frac{1}{2} (\psi^{m+1/2} + \psi^{m-1/2}) \). The constants \( \alpha^{m+1/2} \) and \( \alpha^{m-1/2} \) only depend on the difference scheme and can be determined by the following recurrence relations, which satisfy the case of an isotropic intensity field as described in Ref. [3]:
\[
\alpha^{m+1/2} - \alpha^{m-1/2} = - 2 w^m \mu^m
\]
\[
\alpha^{m+1/2} = \alpha^{m+1/2} = 0
\]  
(14)
Now Eq. (11) can be rewritten as
\[
\frac{\mu^m}{\tau^2} \frac{\partial}{\partial r} (r^2 \psi^m) + \frac{1}{\tau^2 r w^m} \{ \max(\alpha^{m+1/2}) + \max(-\alpha^{m-1/2}) \} \psi^m + \psi^m
\]
\[
= (1-\omega) \theta^m(r) + \frac{\alpha}{2} \sum_{m=1}^{M} w_m \psi^m + \frac{1}{\tau^2 r w^m} \{ \max(-\alpha^{m+1/2}) \} \psi^{m+1}
\]
\[
+ \max(\alpha^{m+1/2}) \psi^m + \psi^m = (1-\omega) \theta^m(r) + \frac{\alpha}{2} \sum_{m=1}^{M} w_m \psi^m
\]
\[
+ \frac{1}{\tau^2 r w^m} \{ \max(-\alpha^{m+1/2}) \} \psi^{m+1} + \max(\alpha^{m-1/2}) \psi^{m-1} \]
\]  
(17)
Rearranging Eq. (17), one gets
\[
\left\{ \frac{\mu^m}{\tau^2} \left( \frac{2}{R_2 - R_1} \right) \frac{\partial}{\partial r} (r^2 \psi^m) + \frac{1}{\tau^2 r w^m} \{ \max(\alpha^{m+1/2}) \} \psi^m + \psi^m = (1-\omega) \theta^m(r) + \frac{\alpha}{2} \sum_{m=1}^{M} w_m \psi^m
\]
\[
+ \frac{1}{\tau^2 r w^m} \{ \max(-\alpha^{m+1/2}) \} \psi^{m+1} + \max(\alpha^{m-1/2}) \psi^{m-1} \}
\]  
(18)
The Gauss–Lobatto collocation points [12–14] are used for spatial discretization
\[
s_i = - \cos \frac{\pi i}{N}, \quad i = 0,1, \ldots,N
\]  
(19)
The Chebyshev approximation of radiative intensity reads
\[
\psi^m_{(i)} = \sum_{k=0}^{N} \hat{W}^m_k T_k(s_i)
\]  
(20)
where the \( T_k(s) \) is the first kind Chebyshev polynomial, and the coefficients \( \hat{W}^m_k, k=0,1,\ldots,N \), are determined by requiring \( \psi^m_{(i)} \) to coincide with \( \psi^m(s) \) at the collocation points \( s_i, i=0,1,\ldots,N \). Therefore, the polynomial of degree \( N \) defined by Eq. (20) is nothing other than the Lagrange interpolation polynomial based on the set \( \{s_i\} \) like
\[
\psi^m_{(s)} = \sum_{j=0}^{N} h_j(s) \psi^m_{(s)}
\]  
(21)
with \( \psi^m_{(s)} = \psi^m(s) \), and \( h_j(s) \) is the polynomial of degree \( N \) defined by
\[
h_j(s) = (-1)^{j+1} (1-s^2) T_j(s) \quad \frac{c_j}{N^2 (s-s_j)}
\]  
(22)
which is a function of the first-order derivative of Chebyshev polynomial. The representation (20) is equivalent to Eq. (21). Thus if expression (21) is substituted into the governing equation one will get the new form whose unknowns are the grid values, and the \( p \)th derivative of \( \psi^m_{(s)} \) with respect to \( s \) can be executed through differentiating \( p \) times directly to the interpolation polynomial (22), namely,
\[
[\psi^m_{(s)}]^{(p)} = \sum_{j=0}^{N} h_j^{(p)}(s) \psi^m_{(s)} = \sum_{j=0}^{N} d_j^{(p)} \psi^m_{(s)}
\]  
(23)
where \( d_j^{(p)} \) are entries of \( p \)th derivative coefficients, which are functions of collocation points only.
For Gauss–Lobatto collocation points (19), the first-order derivative for \( d_j^{(p)} \) reads
\[
d_j^{(1)} = \frac{c_j}{c_j (s_i-s_j)} (-1)^{j+1} \quad 0 \leq i,j \leq N, \quad i \neq j
\]  
(24)
\[ d^{(1)}_{ij} = - \frac{s_i}{2(1 - s_j)}, \quad 1 \leq i \leq N - 1 \]
\[ d^{(1)}_{0i} = - d^{(1)}_{iN} = \frac{2N^2 + 1}{6} \]

and the second-order derivative reads
\[ d^{(2)}_{ij} = \begin{cases} \frac{(N^2 - 1)(1 - s_i)}{s_i^2} + \frac{3}{(s_i - s_j)^2}, & 1 \leq i \leq N - 1 \\ \frac{(N^2 - 1)(1 - s_i)}{s_i^2} - \frac{3}{(s_i + s_j)^2}, & 0 \leq j \leq N, \quad i \neq j \end{cases} \]
\[ d^{(2)}_{k,i} = \begin{cases} \frac{2(-1)^{k+i}}{3} \frac{(N^2 + 1)(1 + s_i) - 6}{(1 + s_i)^2}, & 1 \leq j \leq N \\ \frac{2(-1)^{k+i}}{3} \frac{(N^2 + 1)(1 + s_i) - 6}{(1 + s_i)^2}, & 0 \leq j \leq N - 1 \\ \frac{N^4 - 1}{15} \end{cases} \]

The second-order derivative matrix \( D^{(2)}_N \) can also be computed by \( D^{(2)}_N = D^{(1)}_N + D^{(1)}_N \) directly. Whether for \( D^{(1)}_N \) or \( D^{(2)}_N \), they are needed to be computed once for all in the preparing computation. After substitution of the above derivative matrices, the final discretized form of Eq. (18) reads
\[ \sum_{i=0}^{N} A_{m}^m \psi_i + \sum_{i=0}^{N} B_{m}^m \psi_i = f_m, \quad m = 1, 2, \ldots, M \] (26)

where
\[ A_{m}^m = \begin{cases} \frac{\mu^m}{\tau_{2,m}} \left( \frac{2}{R_{m} - R_{m-1}} \right) D_{i,ik}^{(1)} + \frac{2}{R_{m-1}} \right), & i = k \\ \frac{\mu^m}{\tau_{2,m}} \left( \frac{2}{R_{m} - R_{m-1}} \right) D_{i,ik}^{(1)} \end{cases} \]
otherwise\] (27)

\[ B_{m}^m = \begin{cases} \frac{1}{\tau_{2,m} w^m} \left[ \max(\alpha^{m+1/2}, 0) + \max(-\alpha^{m-1/2}, 0) \right] + 1, & i = k \\ 0 \end{cases} \]
otherwise\] (28)

\[ f_m^m = (1 - \omega) \theta^m(r_i) + \frac{\alpha}{2} \sum_{m=1}^{M} w^m \phi^{m^*}(r_i) \]
\[ + \frac{1}{\tau_{2,m} w^m} \left[ \max(-\alpha^{m+1/2}, 0) \phi^{m_1}(r_i) + \max(\alpha^{m-1/2}, 0) \phi^{m_1}(r_i) \right] \]

Equation (26) is valid for all directions, and for each direction \( m \), the matrices \( A \) and \( B \) have the same size \((0:N,0:N)\) corresponding to the point number \( \{ r_i, i = 0, 1, \ldots, N \} \) and can be incorporated into one matrix. Thus Eq. (26) can be rewritten as
\[ \sum_{i=0}^{N} C_{m}^m \psi_i = f_m^m, \quad m = 1, 2, \ldots, M \] (30)

where \( C_{m}^m = A_{m}^m + B_{m}^m \).

Equation (30) has to be solved with appropriate boundary conditions. One can rewrite Eq. (30) as the following in which the boundary conditions (12a) and (12b) are imported:

\[ \overline{C^m} \overline{\psi} = \overline{F}, \quad m = 1, 2, \ldots, M \] (31)

where
\[ \overline{C} = \begin{cases} C^m(1:N,1:N), & \mu^m > 0 \\ C^m(0:N-1,0:N-1), & \mu^m < 0 \end{cases} \]
\[ \overline{\psi} = \begin{cases} \psi^m(1:N), & \mu^m > 0 \\ \psi^m(0:N-1), & \mu^m < 0 \end{cases} \]
\[ \overline{F} = \begin{cases} f^m(1:N) - C^m(1:N,0) \psi^m(0), & \mu^m > 0 \\ f^m(0:N-1) - C^m(0:N-1,1:N) \psi^m(N), & \mu^m < 0 \end{cases} \] (32)

From the above equations the physical means of the problem should be clearly understood. For positive direction \( \mu^m > 0 \), the radiative intensities on the outer surface of inner sphere \( \phi^m(R_1) \), corresponding to the above vector element \( \overline{\psi}^m(0) \), are computed by Eq. (12a) and should be imported through Eq. (34), but the radiative intensities on the inner surface of outer sphere \( \phi^m(R_2) \), corresponding to the above vector element \( \overline{\psi}^m(N) \), are unknowns. For this situation, a subsquare matrix \( \overline{C} \) with deletion of its first row and first column times the unknown vector \( \overline{\psi} \) with its first element canceled constitutes the left hand side of matrix equation (31). For negative direction \( \mu^m < 0 \), the situation is just reverse.

3.2 Formulations of Diffuse Equation. The discretization of diffuse equation (9) is needed only in space domain \( [R_1,R_2] \). After the mapping of \([R_1,R_2]\) to \([-1,1]\), Eq. (9) becomes
\[ \left( \frac{2}{R_2 - R_1} \right)^2 \frac{d^2 \theta}{d \psi^2} + 2 \left( \frac{2}{R_2 - R_1} \right)^2 \frac{d \theta}{d \psi} = \left( 1 - \omega \right) \left( \frac{\theta^4 - \frac{1}{2} \sum_{m=1}^{M} w^m \phi^m} {N_{tr}} \right) \]

in which the incident radiation \( G(r) \) is replaced by numerical quadrature.

Employing the Chebyshev collocation method for Eq. (35) in the same way as it is for RTE, we obtain the following matrix equation:
\[ \sum_{i=0}^{N} P_{ik} \theta_i = V_i \] (36)

where
\[ P_{ik} = \begin{cases} \left( \frac{2}{R_2 - R_1} \right)^2 D_{i,ik}^{(2)} \left( \frac{2}{R_2 - R_1} \right)^2 D_{i,ik}^{(1)} \right), & i = k \\ \left( \frac{2}{R_2 - R_1} \right)^2 D_{i,ik}^{(2)} + \left( \frac{2}{R_2 - R_1} \right)^2 D_{i,ik}^{(1)} \right), & \text{otherwise} \end{cases} \] (37)

\[ V_i = \left( 1 - \omega \right) \frac{\theta^4(r_i) - \sum_{m=1}^{M} w^m \phi^m(r_i) \right) \]

After the Dirichlet boundary conditions (10) import, Eq. (36) becomes
\[ \overline{P} \overline{\theta} = \overline{V} \] (39)

where
\[ \overline{P} = P(1:N-1,1:N-1) \]
\[ \overline{\theta} = \theta(1:N-1) \] (41)
\[ \bar{V} = V(1: N - 1) - P(1: N - 1, 0) \theta(0) - P(1: N - 1, N) \theta(N) \]  

\[ (42) \]

4 Simultaneous Solution Procedure

From the above formulations we know that the matrix \( \tilde{C}^m \) in Eq. (31) is a function of direction cosine \( \mu, \) node coordinate \( r, \) and angular difference constant \( \alpha. \) If the direction cosine \( \{\mu^1, \mu^2, \ldots, \mu^M\} \) and their corresponding weights \( w^m, \) as well as the collocation points \( \{r_i, i = 0, 1, \ldots, N\}, \) are determined, matrix \( \tilde{C}^m \) needs to be computed once for all in the preparing computation before iteration. Similarly, the matrix \( \tilde{P} \) in Eq. (39) needs to be computed only once.

Because the final matrix equation (31) corresponding to radiative heat transfer contains unknown dimensionless intensity, and the final matrix equation (39) corresponding to conductive heat transfer contains unknown dimensionless radiative intensity, they must be solved iteratively. Finally, the implementation of Chebyshev collocation spectral method for solving coupled radiative and conductive heat transfer can be executed through the following routine.

Step 1. Choose the resolution \( N \) and compute the coordinate values of nodes. Choose the direction number \( M \) and the corresponding direction cosine \( \{\mu^1, \mu^2, \ldots, \mu^M\}, \) as well as the weights \( w^m, \) hence to compute the angular difference constants \( \alpha. \)

Step 2. Preparing compute matrices \( D_c, \tilde{C}^m, \) and \( \tilde{P} \) once for all.

Step 3. Give \( \psi^m \) and \( \theta \) an initial assumption (zero, for example) in the domain of the problem except for boundaries.

Step 4. Using the former solved or initial \( \psi^m \) and \( \theta \) to compute \( \tilde{P}^m \) according to Eq. (29) and compute \( \psi^m(r_i) \) and \( \theta^m(r_i) \) on the boundaries according to Eq. (12) and import the radiative intensity boundary conditions through formula (34).

Step 5. Directly solve Eq. (31) by \( \tilde{P}^m = (\tilde{C}^m)^{-1} \tilde{P}^m \) for all \( m. \)

Step 6. Using the immediately solved \( \psi^m \) and former solved \( \theta \) to compute \( \bar{V} \) according to Eq. (38), import the temperature boundary conditions through formula (42).

Step 7. Directly solve Eq. (39) by \( \theta = \tilde{P}^{-1} \bar{V}. \)

Step 8. Terminate the iteration if the relative maximum absolute difference of dimensionless radiative intensities or temperature, say, \( \max(||(\psi^m(r_i) - (\psi^m(r_i))_0)||/||\psi^m(r_i)||) \) for all nodes and directions, is less than the tolerance \( (10^{-12} \) in our work, for example); otherwise, go back to Step 4.

From the formulations and above steps, the computation tasks mainly exit in the category of linear algebra. The most important two steps special for the matrix equation solution, Steps 5 and 7, can be executed directly and efficiently, while the other steps are concerned with the assembling of matrices only. The inverses \((\tilde{C}^m)^{-1} \) and \((\tilde{P})^{-1}, \) which appear in Steps 5 and 7, are computed using the \( LU \) factorization [37]. In code design, if \( A \) is the inverse of \( B, \) only one code like \( A = \text{inv}(B) \) is needed for the VISUAL FORTRAN system, and \( A = \text{inv}(B) \) is needed for the MATLAB system; otherwise, a small routine is necessary but still efficient.

After solving \( \psi^m \) and \( \theta, \) the dimensionless conductive heat flux \( Q^c, \) the radiative heat flux \( q_r, \) and the total heat flux \( Q \) are determined by the following relations:

\[ Q^c(r) = -\frac{d\theta}{dr}, \quad q_r = 2 \sum_{m=1}^{M} \mu^m w^m \psi^m \]

\[ Q^r(r) = -\frac{d\theta}{dr} + \frac{\tau}{2N_{r,m}} \sum_{m=1}^{M} \mu^m w^m \psi^m \]  

(43)

Table 1 Different orders of ordinate directions and the corresponding weights (positive only)

<table>
<thead>
<tr>
<th>( S )</th>
<th>( S_4 )</th>
<th>( S_6 )</th>
<th>( S_8 )</th>
<th>( S_{10} )</th>
<th>( S_{12} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w^m )</td>
<td>1/2</td>
<td>1/3</td>
<td>1/4</td>
<td>1/5</td>
<td>1/6</td>
</tr>
<tr>
<td>( \mu^m )</td>
<td>0.211325</td>
<td>0.146446</td>
<td>0.102672</td>
<td>0.083752</td>
<td>0.066877</td>
</tr>
<tr>
<td>0.788675</td>
<td>0.500000</td>
<td>0.406205</td>
<td>0.312729</td>
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</tr>
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<td>0.853354</td>
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<td>0.288732</td>
<td>0.288732</td>
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<td>0.614286</td>
<td>0.614286</td>
<td>0.614286</td>
<td></td>
</tr>
</tbody>
</table>

From Eq. (7) one knows that, if the single scattering albedo \( \omega \) is set to be 1, it becomes a pure RTE. As to this situation, one can design a switch in the codes to skip Steps 6 and 7 to solve a radiative heat transfer problem alone.

5 Results and Discussions

In the numerical community, the main superiorities of spectral methods over others like FVM, FEM, and FDM are exponential convergence and high accuracy. In our former work [31,32], the exponential convergence of spectral method for RTE, as well as the ability to capture large oscillations of variables, is confirmed and will not be illustrated again. For example, Fig. 4 in Ref. [32] shows that the accuracy almost keeps the same when the resolution was increased from 5 up to 99 and the direction number from 4 up to 12. For all cases in the present study, we found that the resolution of 11 is enough, and the increase in resolution will not lead to more accurate results obviously. Therefore we used a resolution of 11 for all our computations.

In our present work, the numerical results are presented for two aspects: pure radiation and coupled radiation and conduction, but all in a concentric spherical participating medium.

First, the different order of ordinate directions \( S_{qr} \) for positive direction cosines and the corresponding weights, which are adopted in our computation, are listed in Table 1.

Similar to Ref. [10], the numerical values of \( (r)^2\sigma_q \) for pure radiation under various combinations of \( \varepsilon_1 \) and \( \varepsilon_2 \) are given in Table 2. The symbols SP-DOM, DOM, and FCT in the table mean the results from spectral discrete ordinates method, standard discrete ordinates method, and finite Chebyshev transform, respectively, while the results from DOM and FCT are copied from Ref. [10]. The values in the last column of Table 2 are copied from Ref. [7], which is evaluated by Galerkin method. Comparisons are made for different order of \( S_{qr} \) quadrature numbers in Table 2 clearly shows that our SP-DOM can provide more accurate results than DOM and FCT as a whole. Those numbers in parentheses are relative percentage errors compared with the results from Galerkin method in Ref. [7]. In Table 2, the largest relative error from DOM is 14.55%, from FCT is 13.98%, and from SP-DOM is \(-2.58\%\).

In Fig. 2, the effect of the order of the \( S_{qr} \) quadrature on the radiative heat flux for pure radiation problem is illustrated. Different from FCT in Ref. [10], the \( S_{qr} \) quadrature gives very slight effect on the radiative heat flux in SP-DOM, and the results from \( S_4 \) up to \( S_{10} \) almost show no differences and agree excellently well with those of Ref. [7]. In Fig. 2 only the results from \( S_4 \) and \( S_{12} \) are presented.

In order to show the accuracy of the SP-DOM method, similar to the comparisons in Ref. [10], both nonscattering and scattering solutions are performed for a combined conduction and radiation between two concentric black spherical boundaries. In Table 3, the numerical solutions for the dimensionless total heat flux \( Q^t(r) \) at the boundaries are listed for conduction-radiation parameter \( N_{cr} = 0.1 \) and for various combinations of the optical thickness \( \tau \), the
Table 2: Comparisons of values of \((r^2q_r)\) for various combinations of \(\varepsilon_1\) and \(\varepsilon_2\) with \(R_1=0.5\), 
\(\theta_2=0.5\), and \(\tau_2=1\) (numbers in parentheses are relative errors compared with Ref. [7], in percent)

<table>
<thead>
<tr>
<th>(\varepsilon_1)</th>
<th>(\varepsilon_2)</th>
<th>SP-DOM ((N=4,6,8,10,12))</th>
<th>DOM ([10]) ((N=4,6,8,10,12))</th>
<th>FCT ([10]) ((N=4,6,8,10,12))</th>
<th>Ref. [7]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.20496(25.8)</td>
<td>0.24100(14.55)</td>
<td>0.23980(13.98)</td>
<td>0.21038</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.20680(17.0)</td>
<td>0.22890(8.80)</td>
<td>0.22634(7.59)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.20762(13.1)</td>
<td>0.21317(1.33)</td>
<td>0.22286(8.95)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.20823(1.02)</td>
<td>0.21997(4.56)</td>
<td>0.21938(2.82)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.20846(0.91)</td>
<td>0.21733(3.30)</td>
<td>0.21656(2.94)</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>1</td>
<td>0.10934(1.38)</td>
<td>0.12208(10.11)</td>
<td>0.12148(9.57)</td>
<td>0.11087</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.10986(0.91)</td>
<td>0.11715(5.66)</td>
<td>0.11647(5.05)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.11010(0.69)</td>
<td>0.11587(4.51)</td>
<td>0.11501(3.73)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.11027(0.54)</td>
<td>0.11397(2.80)</td>
<td>0.11385(2.69)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.11035(0.47)</td>
<td>0.11312(2.03)</td>
<td>0.11282(1.76)</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Comparisons of dimensionless total heat flux \(Q_r\) at the inner and outer spheres for various combinations of \(\theta_2\), \(R_1\), and \(\tau_2\) with \(\omega=0\), \(\varepsilon_1=\varepsilon_2=1\), and \(N_\rho=0.1\) (numbers in parentheses are relative errors compared with Ref. [5], in percent)

<table>
<thead>
<tr>
<th>(\theta_2)</th>
<th>(R_1)</th>
<th>(\tau_2)</th>
<th>SP-DOM</th>
<th>DOM ([10])</th>
<th>FCT ([10])</th>
<th>Ref. [5]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.5</td>
<td>2</td>
<td>7.603(−1.02)</td>
<td>7.742(0.79)</td>
<td>7.720(0.51)</td>
<td>7.681</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>5.994(−1.37)</td>
<td>6.130(0.89)</td>
<td>6.111(0.56)</td>
<td>6.077</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>9.866(−0.33)</td>
<td>9.945(0.46)</td>
<td>9.921(0.22)</td>
<td>9.899</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>11.259(−1.71)</td>
<td>11.475(0.17)</td>
<td>11.462(0.06)</td>
<td>11.455</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2.468(−0.08)</td>
<td>2.498(1.13)</td>
<td>2.493(0.93)</td>
<td>2.470</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.914(−1.59)</td>
<td>1.965(1.03)</td>
<td>1.958(0.67)</td>
<td>1.945</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>10.614(−0.83)</td>
<td>10.942(2.21)</td>
<td>11.037(3.10)</td>
<td>10.705</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.091(0.99)</td>
<td>1.035(0.78)</td>
<td>1.030(0.52)</td>
<td>1.020</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.499(−1.32)</td>
<td>1.532(0.86)</td>
<td>1.528(0.59)</td>
<td>1.519</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>7.991(−0.27)</td>
<td>8.055(0.52)</td>
<td>8.036(0.29)</td>
<td>8.013</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2.832(−1.12)</td>
<td>2.868(0.14)</td>
<td>2.865(0.03)</td>
<td>2.864</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.617(5.11)</td>
<td>0.625(6.47)</td>
<td>0.623(6.13)</td>
<td>0.587</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.478(−1.65)</td>
<td>0.492(1.23)</td>
<td>0.489(0.62)</td>
<td>0.486</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.106(−0.93)</td>
<td>0.109(1.87)</td>
<td>0.110(2.80)</td>
<td>0.107</td>
</tr>
</tbody>
</table>

ratio \(R_1/R_2\), and the ratio of boundary temperature \(\theta_2\). The results indicate that our SP-DOM solutions with \(S_{12}\) quadrature also agree well with those of Viskanta and Merriam [5] and better than those of DOM and FCT. Compared with the results from Ref. [5], as to the total heat flux \(Q_r(R_1)\), the largest relative error from DOM is 2.21\%, from FCT is 3.10\%, and from SP-DOM is \(-1.71\%\); as to \(Q_r(R_1)\), the largest relative error from DOM is 6.47\%, from FCT is 6.13\%, and from SP-DOM is 5.11\%.

The numerical values of dimensionless conductive \(Q_r(r)\) and total heat flux \(Q_r(r)\) at the boundaries for scattering medium are listed in Table 4. The results in Table 4 show that the SP-DOM can give very good solutions referring to those of Jia et al. [7]. This time, the largest relative error from DOM is 2.46\%, from FCT is \(-1.53\%\), but from SP-DOM is up to 2.51\%. However, all these three largest errors are less than 3\%.

To further prove the accuracy of the SP-DOM method, more comparisons between our results with those available in Ref. [5] are made. In Table 5, the dimensionless conductive heat flux \(Q_r\) total heat flux \(Q_r\), radiative heat flux \(q_r\) at the inner and outer spheres for various temperature ratios with \(\theta_1=1.0\), \(\tau_2=2\), \(R_1=0.5\), \(\varepsilon_1=\varepsilon_2=1\), and \(N_\rho=0.1\) are listed for SP-DOM and Ref. [5].

In Table 6, predictions of dimensionless total heat flux at the inner sphere with \(\theta_1=1.0\) and \(\theta_2=0.5\) by SP-DOM and Ref. [5] are...
Table 4 Comparisons of dimensionless conductive heat flux $Q_c$ and total heat flux $Q_t$ at the inner and outer spheres for various combinations of $N_{cr}$ and $\omega$ with $R_1=0.5$, $\epsilon_1=\epsilon_2=1$, $\theta_2=0.5$, and $\gamma_2=2$ (numbers in parentheses are relative errors compared with Ref. [7], in percent)

<table>
<thead>
<tr>
<th>$N_{cr}$ (SP-DOM, DOM [10], FCT [10], Ref. [7])</th>
<th>$\omega$</th>
<th>$Q_c(R_1)$</th>
<th>$Q_c(R_2)$</th>
<th>$Q_t(R_1)$</th>
<th>$Q_t(R_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9</td>
<td>2.0062(−0.05)</td>
<td>2.0064(−0.04)</td>
<td>2.0073</td>
<td>2.0312(−0.13)</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>2.0303(−0.18)</td>
<td>2.0606(1.31)</td>
<td>2.039</td>
<td>2.0312(−0.13)</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>2.0511(−0.24)</td>
<td>2.0629(0.34)</td>
<td>2.0527(−0.16)</td>
<td>2.0560</td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>2.0628(−0.48)</td>
<td>2.0647(−0.39)</td>
<td>2.0728</td>
<td>2.3323</td>
</tr>
</tbody>
</table>

listed and compared with those exact values. The results in these two comparable tables illustrate the good accuracy of SP-DOM again.

The authors would like to declare that Eq. (39), which is originally discretized from Eq. (35), cannot provide smooth solutions for all cases listed in Tables 3–5. The main reason may be the characteristic of the strong nonlinearity of Eq. (35).

To reduce the nonlinearity of Eq. (35), we first rewrite it to a new equivalent form

$$\frac{d^2 \theta}{dr^2} + \frac{2d \theta}{r dr} + \frac{2(1 - \omega) \gamma_2}{N_{cr}} \theta^2 = -\frac{(1 - \omega) \gamma_2^2}{2N_{cr}} \sum_{m=1}^{M} \omega_m \Psi_m - \frac{(1 - \omega) \gamma_2^2}{N_{cr}} \theta^2$$

and then arrange Eq. (44) as

Table 5 Comparisons of dimensionless conductive heat flux $Q_c$, total heat flux $Q_t$, and radiative heat flux $q_r$ at the inner and outer spheres for various temperature ratios with $\theta_1=1.0$, $\gamma_2=2$, $R_1=0.5$, $\epsilon_1=\epsilon_2=1$, and $N_{cr}=0.1$

<table>
<thead>
<tr>
<th>$\theta_2$ (SP-DOM, Ref. [5])</th>
<th>$Q_c(R_1)$</th>
<th>$Q_c(R_2)$</th>
<th>$q_r(R_1)$</th>
<th>$Q_t(R_1)$</th>
<th>$Q_t(R_2)$</th>
<th>$q_r(R_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>3.565</td>
<td>7.603</td>
<td>0.808</td>
<td>1.327</td>
<td>1.901</td>
<td>0.115</td>
</tr>
<tr>
<td>0.25</td>
<td>3.607</td>
<td>7.679</td>
<td>0.814</td>
<td>1.319</td>
<td>1.920</td>
<td>0.120</td>
</tr>
<tr>
<td>0.5</td>
<td>3.191</td>
<td>7.900</td>
<td>0.780</td>
<td>1.122</td>
<td>1.772</td>
<td>0.130</td>
</tr>
<tr>
<td>0.75</td>
<td>3.233</td>
<td>7.994</td>
<td>0.790</td>
<td>1.114</td>
<td>1.792</td>
<td>0.136</td>
</tr>
<tr>
<td>0.9</td>
<td>2.521</td>
<td>6.076</td>
<td>0.733</td>
<td>0.724</td>
<td>1.519</td>
<td>0.159</td>
</tr>
<tr>
<td>0.75</td>
<td>1.559</td>
<td>3.984</td>
<td>0.485</td>
<td>0.341</td>
<td>0.996</td>
<td>0.131</td>
</tr>
<tr>
<td>0.9</td>
<td>1.585</td>
<td>4.046</td>
<td>0.492</td>
<td>0.337</td>
<td>1.012</td>
<td>0.135</td>
</tr>
<tr>
<td>0.9</td>
<td>0.713</td>
<td>1.914</td>
<td>0.240</td>
<td>0.131</td>
<td>0.479</td>
<td>0.070</td>
</tr>
<tr>
<td>0.9</td>
<td>0.726</td>
<td>1.945</td>
<td>0.244</td>
<td>0.129</td>
<td>0.486</td>
<td>0.071</td>
</tr>
</tbody>
</table>


