Computational Efficiency of the Finite Element Method based on Second-Order Radiative Transfer Equation

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Numerical Insight into the RTE

First-Order Radiative Transfer Equation (FORTE, RTE):

\[ \Omega \cdot \nabla I + (\kappa_a + \kappa_s)I = \kappa_a I_b + \frac{\kappa_s}{4\pi} \int I(r, \Omega)\Phi(\Omega, \Omega')d\Omega \]

Boundary condition:

\[ I(r_w, \Omega) = \overline{I}_{bw}(r_w), \quad n_w \cdot \Omega > 0 \]
Numerical Insight into the RTE

\[ \Omega \cdot \nabla I + \beta I = \kappa_a I_b + \frac{\kappa_s}{4\pi} \int I(\mathbf{r}, \Omega') \Phi(\Omega', \Omega) d\Omega' \]

- If take direction as "velocity", what it look like?

\[ \nabla \cdot \nabla I + \beta I = S \]
Numerical Insight into the RTE

\[ \mathbf{\Omega} \cdot \nabla I + \beta I = \kappa_a I_b + \frac{\kappa_s}{4\pi} \int \mathbf{I}(r, \Omega') \Phi(\Omega', \Omega) \, d\Omega' \]

- If take direction as “velocity”, what it look like?

\[ \mathbf{V} \cdot \nabla I + \alpha \nabla^2 I + \beta I = S \]
Numerical Insight into the RTE

\[ \Omega \cdot \nabla I + \beta I = \kappa_a I_b + \frac{\kappa_s}{4\pi} \int I(r, \Omega') \Phi(\Omega', \Omega) d\Omega' \]

- If take direction as “velocity”, what it look like?

\[ \nabla \cdot \nabla I + \alpha \nabla^2 I + \beta I = S \]

- Due to the absence of diffusion term, it is a Convection-dominated type of general convection diffusion equation
Difficulties in Numerical Solving RTE

1) Stability problem

- Caused by Convection-dominated characteristics of RTE
Difficulties in Numerical Solving RTE

1) **Stability problem**
   - Caused by *Convection-dominated characteristics* of RTE

2) **False Scattering**
   - Num. Phenomena: False energy diffusion
   - Cause: Insufficient spatial accuracy
Difficulties in Numerical Solving RTE

1) Stability problem
   - Caused by Convection-dominated characteristics of RTE

2) False Scattering
   - Num. Phenomena: False energy scattering
   - Cause: Insufficient spatial accuracy

3) “Ray Effects”
   - Num. Phenomena: nonphysical wiggles in results
   - Cause: Insufficient angular quadrature accuracy, may coupled with 1) and 2)
Example of Causes of Ray effects

Boundary load with large nonuniformity
Example of Causes of Ray effects

Boundary load with large nonuniformity

Interior Obstacle shielding
Example of Causes of **Ray effects**

**Boundary load with large nonuniformity**

**Interior Obstacle shielding**

**Large gradient of Source**
Example of Causes of Ray effects

Boundary load with large nonuniformity

Interior Obstacle shielding

Large gradient of Source

Zhao & Liu (JQSRT, 2007)

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Solution to Difficulties

1) **Stability problem**
   - Basic rationale
     - (A) Upwinding in discretization, artificial diffusion ...
     - (B) Transform FORTE to second order equation, or cancel the convection term
   - Even-Parity formulation of RTE (EPF-RTE)
   - Second Order RTE (SORTE)

2) **False Scattering**

3) **“Ray Effects”**
Advantage of Solution (B)

- No artificial diffusion is needed to be intentionally added.
- Based on the equation, radiative transfer can be solved stably with many methods, FEM, FVM, Meshless method,
- Hence it is a unified approach, one for all.
Disadvantage of EPF-RTE

- Solution variable is not radiative intensity
- Difficult to be extended to anisotropic scattering media
Introduction to SORTE

Derivation

\[
\frac{d}{ds} I + \beta I = \kappa_a I_b + \frac{\kappa_s}{4\pi} \int_S I(r, \Omega') \Phi(\Omega', \Omega) d\Omega'
\]
Introduction to SORTE

Derivation

\[
\frac{1}{\beta} \frac{d}{ds} I + I = \frac{1}{\beta} S
\]
Introduction to SORTE

- **Derivation**

\[
\frac{d}{ds}\left[\frac{1}{\beta} \frac{d}{ds} I\right] + I = \frac{1}{\beta} S
\]

\[
\frac{d}{ds}\left[\frac{1}{\beta} I\right] + \frac{d}{ds} I = \frac{d}{ds}\left[\frac{1}{\beta} S\right]
\]

\[
- \frac{d}{ds}\left[\frac{1}{\beta} I\right] + \beta I = S - \frac{d}{ds}\left[\frac{1}{\beta} S\right]
\]
Introduction to SORTE

Second Order Radiative Transfer Equation (SORTE)\([\text{Zhao & Liu}(2007)]\):

\[-\beta^{-1}\Omega \cdot \nabla \left[ \beta^{-1}\Omega \cdot \nabla I \right] + I = S - \beta^{-1}\Omega \cdot \nabla S\]

Properties of the SORTE

- Convection term is cancelled and replaced by a diffusion term
- Solution variable is intensity
- Can be easily applied to anisotropic scattering media, without limit on general applicability of FORTE
Boundary conditions

\[ \mathbf{n}_w \cdot \Omega \leq 0 \]

Outflow

\[ \Gamma = \Gamma_D \cup \Gamma_N \]

Inflow

\[ \mathbf{n}_w \cdot \Omega > 0 \]
Solved intensity distribution in a slab with a Gaussian hill source

\[ \text{[Zhao & Liu(2007)]} \]
Then, However,

- **What is the weakness of the SORTES?**
  - Probably the most important is the computational efficiency of the numerical methods based on it

- **Which is crucial for broad application of this approach.**

- **As such, this subject forms the major motivation of present research**
Objectives of this work

Investigate the accuracy and solution cost of finite element method (FEM) based on the SORTE
Formulation and Implementation

**SORTE in 2D can be written as:**

\[
(\mu^m)^2 \frac{\partial^2 I^m}{\partial x^2} + (\eta^m)^2 \frac{\partial^2 I^m}{\partial y^2} + 2\mu^m\eta^m \frac{\partial^2 I^m}{\partial x \partial y} - \beta^2 I^m = U^m
\]

\[
U^m = \Omega^m \cdot \nabla S^m - \beta S^m
\]
Formulation and Implementation

**FEM discretization of the SORTE:**

1) *FEM* approximation

\[ I^m(r) \approx \sum_{i=1}^{N_{sol}} I_i^m \phi_i(r) \]

2) *Galerkin* approach

\[ \sum_{i=1}^{N_{sol}} I_i^m \int_V \left[ (\mu^m)^2 \frac{\partial^2 I^m}{\partial x^2} + (\eta^m)^2 \frac{\partial^2 I^m}{\partial y^2} + 2\mu^m \eta^m \frac{\partial^2 I^m}{\partial x \partial y} - \beta^2 I^m \right] \phi_j(r) dV \]

\[ = \int_V U^m(r) \phi_j(r) dV \]

3) final matrix form: \[ K^m I^m = H^m \]
By using tool matrices approach [Zhao & Liu (2006)]:

\[ K^m = (\mu^m)^2 A^{xx} + \mu^m \eta^m A^{xy} + \eta^m \mu^m (A^{xy})^T + (\eta^m)^2 A^{yy} + \beta^2 B^{oo} + \beta \left( \mu^m N^x + \eta^m N^y \right) \]

\[ H^m = \left[ \beta B^{oo} - \mu^m (B^{xo})^T - \eta^m (B^{yo})^T + \beta \left( \mu^m N^x + \eta^m N^y \right) \right] S^m \]

\[ A_{ji}^{xx} = \int_V \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} dV \quad A_{ji}^{xy} = \int_V \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial y} dV \quad A_{jn}^{yy} = \int_V \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial y} dV \]

\[ B_{ji}^{xo} = \int_V \frac{\partial \phi_j}{\partial x} \phi_i dV \quad B_{jn}^{yo} = \int_V \frac{\partial \phi_j}{\partial y} \phi_i dV \quad B_{jn}^{oo} = \int_V \phi_j \phi_i dV \]

\[ N_{ji}^x = \int_{\Gamma_N} \phi_j \phi_i (n_w \cdot \mathbf{i}) dA \quad N_{ji}^y = \int_{\Gamma_N} \phi_j \phi_i (n_w \cdot \mathbf{j}) dA \]

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FEM discretization of the FORTE:

1) **FEM** approximation

$$I^m(r) \approx \sum_{i=1}^{N_{sol}} I_i^m \phi_i(r)$$

2) *Galerkin* and *Least Squares* approach

3) final matrix form:

$$K^m I^m = H^m$$
FORTE with Galerkin scheme:

\[ K^m = \mu^m (B^{xo})^T + \eta^m (B^{yo})^T + \beta B^{oo} \]

\[ H^m = B^{oo} S^m \]
Formulation and Implementation

FORTE with Least square scheme:

\[
K^m = (\mu^m)^2 A^{xx} + \mu^m \eta^m A^{xy} + \mu^m \beta B^{xo} \\
+ \eta^m \mu^m (A^{xy})^T + (\eta^m)^2 A^{yy} + \eta^m \beta B^{yo} \\
+ \beta \mu^m (B^{xo})^T + \beta \eta^m (B^{yo})^T + (\beta)^2 B^{oo} \\
H^m = \left( \mu^m B^{xo} + \eta^m B^{yo} + \varsigma^m B^{zo} + \beta B^{oo} \right) S^m
\]
Generic Solution procedures

1. Input: mesh, etc.
2. Calculate tool matrices: A, B, ...
3. Global iteration, for n = 1 to Max. iter.
4. Angular iteration, for j = 1 to M
5. Assemble stiff matrices: \( K^m, H^m \)
6. Solve equation
7. Is angular iteration finished?
   - No
   - Yes: Output results
8. Is converged?
   - No
   - Yes: Output results
   - \( \frac{|G_{old} - G_{new}|}{|G_{new}|} < \varepsilon \)
9. Impose boundary condition
10. From left
11. To right
Results and Discussion

Case 1: Semicircular enclosure with a circular hole

Configuration of the semicircular enclosure and mesh decomposition (272 elements).

\[ \tau_L = \beta R = 0.1 \]

\[ T_g = 1000 \text{ K} \]
Results and Discussion

Space: 272 elements, shape function is constructed through 3rd order Chebyshev approximation,

Solid angle:

\[ N_\theta \times N_\phi = 20 \times 40 \]

Heat flux distribution along bottom wall

\[ \frac{q_{wl}}{\sigma T^4_g} \]
Results and Discussion

Galerkin-FORTE

LS-FORTE  Galerkin-SORTE

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Results and Discussion

Case 2: Isotropically Scattering Medium in a Square Enclosure

\[ \omega = 1 \]
\[ \tau_L = \beta L = 1 \]
\[ T_g = 0 \text{ K} \]

Space: \( M \times M \) bilinear elements, \( M \) is taken as needed for convergence analysis.

Solid angle: \( S_8 \)

Solution quantity: bottom wall radiative heat flux

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Results and Discussion

(a) $\tau_L = 0.25$

- LS-FORTE
- Galerkin-FORTE
- Galerkin-SORTE

Max. Relative Error, %

Computational Time, s

$N_{sol}$
Results and Discussion

(b) $\tau_L = 1$

\[ N_{sol} \]

Max. Relative Error, %

- LS-FORTE
- Galerkin-FORTE
- Galerkin-SORTE

Computational Time, s
Results and Discussion

(c) $\tau_L = 10$

- LS-FORTE
- Galerkin-FORTE
- Galerkin-SORTE

Max. Relative Error, %

Computational Time, s

$N_{sol}$
Conclusions

- The accuracy of the FEM based on the SORTE is generally better than that based on the FORTE.
- FEM based on SORTE is the most efficient than the FEMs based on the FORTE.
Questions & comments?

Thanks for your attention!