Disconnected-connected network transitions and phase separation driven by coevolving dynamics

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China-Europe Summer School on Complexity Science
(10-14 August 2010, Shanghai)
Dynamic models (SIS, SIR, opinion formation), or games (PD, SG, …)

NETWORKS (group dynamics)

Two dynamics influencing one another

COEVOLVING SYSTEM

NEW FEATURES?

COMPUTER SIMULATIONS

THEORIES

REAL SYSTEMS

To read more on the topic in general:
Perc and Szolnoki, Biosystems 99, 109 (2009)
Szabo and Fath, Physics Reports 446, 97 (2007)
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The general ideas have been applied to: 
Adaptive epidemic models: e.g., Gross et al., PRL 96, 208701 (2006); Shaw and Schwartz, PRE 71, 066101 (2008) 
Opinion formation models: e.g., Vazquez et al., PRL 100, 108702 (2008); Nardini et al., PRL 100, 158701 (2008) 
Wars and human conflicts: e.g. Bohorquez et al., Nature 462, 911 (2009); Zhao et al., PRL 103, 148701 (2009)
And more…(from PM Hui’s group):
Modeling of guilds in online games (World of Warcraft) and LA street gangs
-- Zhao et al., PRE 79, 066117 (2009)

Effects of social group dynamics on contagion (YouTube downloads, foreign exchange rates, flu) – Zhao et al., PRE 81, 056107 (2010)
In this summer school...

• use an adaptive snowdrift game as an example to show

-- how coupled dynamics influence each other – disconnected-connected network transition (network structure) accompanying a segregation-mixed phase transition (dynamic model)

-- how one could approach such problems analytically

-- what to look at in formulating a theory
**Snowdrift Game (SDG) [1]**

- **Scenario:**
  - Two drivers heading home in opposite directions
  - Blocked by a snowdrift
  - Each driver: 2 actions/characters
    - **C** ("cooperate") = to shovel the snowdrift
    - **D** ("not-to-operate") OR "defect" (in prisoner’s dilemma language) = not to shovel

b = reward of getting home

\( c = \text{cost (doing the laborious job of shoveling)} \) \quad b > c > 0

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>C</strong></td>
<td>( \left( \frac{b-c}{2}, \frac{b-c}{2} \right) )</td>
<td>( (b-c, b) )</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>( (b, b-c) )</td>
<td>( (0, 0) )</td>
</tr>
</tbody>
</table>

- **Player 1**
- **Player 2**

Sucker payoff
<table>
<thead>
<tr>
<th>Player 1</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>( \left( \frac{b-c}{2}, \frac{b-c}{2} \right) )</td>
<td>( (b-c, b) )</td>
</tr>
<tr>
<td></td>
<td>( (R, R) )</td>
<td>( (S, T) )</td>
</tr>
<tr>
<td>D</td>
<td>( (b, b-c) )</td>
<td>( (0, 0) )</td>
</tr>
<tr>
<td></td>
<td>( (T, S) )</td>
<td>( (P, P) )</td>
</tr>
</tbody>
</table>

- \( b > c > 0 \) defines the snowdrift game
- It follows that \( T > R > S > P \) (defines SDG)
Showing only the payoffs of player 1:

<table>
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<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>R</td>
<td>S</td>
</tr>
<tr>
<td>D</td>
<td>T</td>
<td>P</td>
</tr>
</tbody>
</table>

- **Snowdrift Game:**
  \[ T > R > S > P \]

- **Prisoner’s Dilemma:**
  \[ T > R > P > S \]

Difficult to measure payoffs accurately

⇒ SDG is an alternative to PD in studying cooperation in competing populations
Using one parameter $r$ to represent the payoffs:

\[ T > R > S > P \ ( = 0 ) \]

\[ 1 + r > 1 > 1 - r > 0 \quad (0 < r < 1) \]

\[
\begin{array}{c|cc}
\text{cost} & C & D \\
\hline
\text{reward} & 1 & 1+r \\
C & 1 & 1-r \\
D & 1+\text{r} & 0 \\
\end{array}
\]

$r = \frac{\text{cost}}{\text{reward}}$ (larger $r$ tends to promote D-character)
How to assign switching probabilities (CS and DS events)?

Dissatisfaction! Doesn’t meet expectation, thus rational!
Expectation: When I play C, I expect to get 1 (opponent is C)
               When I play D, I expect to get 1+r (opponent is C)

Thus, when opponent is C, received = expected
               => no incentive to make any changes

Thus, dissatisfaction comes in only when opponent plays D
               => switch character or rewiring

We define a parameter, called the disappointment $S$, when opponent plays D as

$$S = \text{expected payoff} - \text{received payoff} = P(\alpha, C) - P(\alpha, D)$$

**Switching Probability** $P \propto S \rightarrow P = \beta \cdot S$  (Here, we take $\beta = 1/2$)

If not switched, cut link and rewire to someone else.

Node-driven dynamics

CD-links AND DD-links are the active links (possible system evolution)
Probabilities for the 4 events that lead to system evolution

\[
\begin{align*}
    P_{D,\text{rewire}} &= \frac{1+r}{2} \\
    P_{C,\text{rewire}} &= 1 - \frac{r}{2} \\
    P_{D,\text{switch}} &= \frac{1-r}{2} \\
    P_{C,\text{switch}} &= \frac{r}{2}
\end{align*}
\]
How does the level of cooperation (long time behavior) vary with r?

How does dissatisfaction behavior alter network structure?

Time evolution?

Constructing analytic approaches?
Initially, we have 50% cooperators randomly distributed in the lattice.

The results indicate two regimes with different features.

What if the initial frequency of cooperators is varied?

Definition:
\[ f_c = \frac{\text{number of C-nodes}}{\text{number of total nodes}} \]

This figure gives us a message of the extent of cooperation.

But how are the different characters connected?
• Initially, we have 10% cooperators randomly distributed in the lattice.

• Definition:
  \[ l_{xy} = \frac{\text{number of } XY\text{-links}}{\text{number of total links}} \]

• The symbols show a transition behavior at some value of \( r \).

• Can we obtain the features of the previous figure based on the link densities?

◆ This figure gives us a message of link distributions on network.
Disconnected-connected network transition accompanying a C/D phase-separated and mixed phase transition

Ref: Graser, Xu, Hui, EPL 87, 38003 (2009)
Trajectory of Systems

(a) & (b): Initial frequency of cooperation $f_c^i = 0.1$

(c) & (d): For cost-to-benefit ratio $r = 0.9$

- More simulation results — trajectories showing time evolution

- Definition (x-axis):

$$m = \frac{N_C - N_D}{N}$$

(a) & (b): Initial frequency of cooperation $f_c^i = 0.1$

(c) & (d): For cost-to-benefit ratio $r = 0.3$. 
Constructing a theory…

Recall: The number of Cooperators (i.e., $N_C$ or $f_C$ or $m$) determines the fraction of characters on nodes.

The number of links (i.e., $L_{CC}$ or $l_{CC}$ or $m_l$) indicates, on average, how links are distributed between nodes.

When an event occurs, local environment of updating node changes, leading to corresponding changes in variables such as node/link numbers.
Write down the change of the variables in general (recall: CD and DD are active links):

\[ \Delta X = \sum_{n=C,D} P_n \sum_{\kappa} P_{n,\kappa} \sum_{\lambda_{nd}=0} P_{\lambda_{nd}} \frac{\lambda_{nd}}{\kappa} \sum_{E=CS,DS} \sum_{CR,DR} P_E \Delta X(\kappa, \lambda_{nd}) \] (*)

- **Probability that the node in action takes on character** \( n = C \) or \( D \)
- **Probability that among** \( \kappa \) **links there are** \( \lambda_{nd} \) **links**
- **Fraction of** \( nd \)-**links among** \( \kappa \) **links (prob. of picking an active link)**
- **Conditional probability of having** \( \kappa \) **links around a node of character** \( n \)
- **Probability that an event** \( E \) **occurs (switch/rewire) under the condition that the node is of character** \( n \)

**Remark:** Could start from node-level equations and construct equations for global quantities
Definitions: \( M = N_C - N_D \), \( M_l = L_{CC} - L_{DD} \)

<table>
<thead>
<tr>
<th></th>
<th>( C \to D )</th>
<th>( D \to C )</th>
<th>( CD)-rewire</th>
<th>( DD)-rewire</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta M )</td>
<td>-2</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \Delta L_{CD} )</td>
<td>( \lambda_{CC} - \lambda_{CD} )</td>
<td>( \lambda_{DD} - \lambda_{CD} )</td>
<td>( -1 + N_D / N )</td>
<td>( N_C / N )</td>
</tr>
<tr>
<td>( \Delta M_l )</td>
<td>( - \lambda_{CC} - \lambda_{CD} )</td>
<td>( \lambda_{DD} + \lambda_{CD} )</td>
<td>( N_C / N )</td>
<td>( N_C / N )</td>
</tr>
</tbody>
</table>

![Diagram](image_url)
To illustrate the different ways of mean-field treatment, we write down one of the equations based on (\(*\)):

\[
\Delta L_{CC} = f_c P_C^{(S)} \left( \frac{\lambda_{CD}^2}{\kappa} \right)_C - \langle \lambda_{CD} \rangle_C + f_c^2 P_C^{(R)} \left( \frac{\lambda_{CD}}{\kappa} \right)_C + (1 - f_c) P_D^{(S)} \left( \lambda_{CD} \right)_D - \left( \frac{\lambda_{CD}^2}{\kappa} \right)_D
\]

Here, we denote \( \left\langle \frac{\lambda_{CD}^2}{\kappa} \right\rangle \) as \( \langle \lambda_{CD} \rangle \) variables.

\( \langle \cdots \rangle \) denotes an average over a type of nodes (subscript).

Note: These averages distinguish different types of nodes. Need to treat C and D nodes separately.
To proceed, we first decouple the quantity of the form

\[ \left\langle \frac{\lambda^n_{CD}}{\kappa} \right\rangle_{C/D} = \frac{\langle \lambda^n_{CD} \rangle_{C/D}}{\langle \kappa \rangle_{C/D}} \]

We treat the first moments using global mean values, i.e.,

\[ \langle \lambda_{CD} \rangle_C = \frac{l_{CD}}{f_c} \]

\[ \langle \lambda_{CD} \rangle_D = \frac{l_{CD}}{1 - f_c} \]

\[ \langle \kappa \rangle_C = \frac{2l_{CC} + l_{CD}}{f_c} \]

\[ \langle \kappa \rangle_D = \frac{k - 2l_{CC} - l_{CD}}{1 - f_c} \]
• Aim: To close the set of equations.
• There are several ways to treat the second moments.

(1) **Simple Squared Closure (SSC) [simplest approximation]**

\[
\langle \lambda_{CD}^2 \rangle_{(SSC)}^D = \langle \lambda_{CD}^2 \rangle_{(SSC)}^C = \frac{l_{CD}^2}{f_c^2}
\]

\[
\langle \lambda_{CD}^2 \rangle_{(SSC)}^D = \langle \lambda_{CD}^2 \rangle_{(SSC)}^D = \frac{l_{CD}^2}{(1 - f_c)^2}
\]

Second moments assumed to be equal to the first moments squared.

Physical Meaning: Every C node has identical neighborhood, every D node has identical neighborhood; but C and D could have different neighborhoods (thus ignored fluctuations).
A closed set of 3 equations – does it work?

Lines: From closed set of equations using SSC. Capture all key features, including non-monotonic behavior!

Ref: Graser, Xu, Hui, EPL 87, 38003 (2009)
Theory captures the disconnected-connected network transition and phase separation.
How about the trajectories?

With initial frequency of cooperation $f_c = 0$.

Line shows the locations of endpoints as calculated by iterating closed set of equations to long time.
Time evolution of degrees

Steady state mean degrees

$fc^i=0.8$ and $r=0.8$

$fc^i=0.1$

Lines are results of equations, symbols are simulation results
While simple theory captures all the main features, there are discrepancies between simple theory and numerical results!

Can we improve the theory? Better moment closure schemes?
• Alternative ways to treat the second moments and to close the equations

(2) Binomial distribution treatment (BINO)

$$\langle \lambda_{CD}^2 \rangle_{C}^{(BINO)} = \frac{l_{CD}^2}{f_c^2} + \frac{l_{CD}}{f_c} \left( \frac{l_{CD}^2}{f_c} - \frac{l_{CD}^2}{f_c(2l_{CC} + l_{CD})} \right)$$

$$\langle \lambda_{CD}^2 \rangle_{D}^{(BINO)} = \frac{l_{CD}^2}{(1-f_c)^2} + \frac{l_{CD}}{1-f_c} \left( \frac{l_{CD}^2}{1-f_c} - \frac{l_{CD}^2}{(1-f_c)(\kappa - 2l_{CC} - l_{CD})} \right)$$

Comparing with SSC treatment, we have extra terms in BINO treatment!

Picture: Assume that C-node (or D-node) of degree \( \kappa \) has a binomial distribution of links \( \lambda_{CD} \).
(3) Aside: *Keeling-Eames treatment* (KE)


Comparing with SSC treatment, there is an extra term in KE treatment.

\[
\langle \lambda_{CD}^2 \rangle^{(KE)}_{C/D} = \langle \lambda_{CD} \rangle^2_{C/D} + \langle \lambda_{CD} \rangle_{C/D}
\]

\[
\sigma^2 = \langle \lambda_{CD}^2 \rangle - \langle \lambda_{CD} \rangle^2 = \langle \lambda_{CD} \rangle = \mu
\]

Fluctuations are included in a way that assumes the variance equals to the mean.

But KE approximation turns out to be a bad approximation for the present model.
Comparing simple and modified theories with numerical results

SSC treatment (dashed lines)

BINO treatment (solid lines) – Worked much better in the disconnected state

Ref: Oliver Graser, PhD Dissertation (CUHK 2010)
Final Remarks

- Used an adaptive snowdrift game to illustrate the richness in co-evolving dynamical systems and the approaches to analytic formalism

- Approach that works in one problem (e.g., SIS) may not work in another (SG)

- (No shown) Interesting results follow from fixed-point analysis based on closed set of equations (disconnected state as a fixed point competing with a connected state with a shrinking attractive basin)