A rigorous derivation of the Lorentz transformation
based on minimum assumptions

Xing-Bin Huang*

College of Physics Science and Technology, Heilongjiang University, Harbin 150080, People’s Republic of China

PACS: 03.30.+p – Special relativity

Abstract. Besides Einstein’s two fundamental postulates, additional assumptions are made in various derivations of the Lorentz transformation (LT). Two or three so-called exact derivations of the LT have been highly abstract and abstruse. Thus, the validity of the derivation of the LT has been questioned. Here, we present a rigorous derivation of the LT, the approach of which differs from that typically employed. Our derivation is only based on the constancy of the speed of light and two thought experiments. No additional assumptions are needed because the constancy of the speed of light is used to prove the necessary assumptions.

1. Introduction

Einstein established his remarkable special theory of relativity in 1905. Einstein derived the Lorentz transformation (LT), which is the central part of his theory, from two simple postulates. However, Einstein made additional assumptions in deriving the LT [1, 2]. Therefore, his derivation is very difficult to accept absolutely, and various derivations of the LT have been published in relativistic literature [3–7]. Nonetheless, in the present derivations, the additional assumptions are not entirely abandoned. Here we list Einstein’s two postulates and common additional assumptions.

- **Postulate of relativity:** The laws of physics are the same in all inertial frames of reference.
- **Postulate of the constancy of the speed of light:** The speed of light in a vacuum is the same in all inertial frames of reference and is independent of the motion of the source.
- **Assumption** of the invariance of relative velocity between two inertial coordinates. If Mary sees Henry is moving away from her with a constant velocity \( u \), then Henry sees Mary moving away from him with the same velocity \( u \).
- **Assumption** that lengths measured perpendicular to the relative motion of the two frames are not contracted; i.e., \( y \) and \( z \) coordinates transform in a Galilean manner.
- **Assumption** that relations of the coordinate transformation are linear.
- **Assumption** that space–time is homogeneous and isotropic.

Physicists studying the derivation of the LT are still attempting to eliminate the additional assumptions [8]. In particular, the first two assumptions are often used; i.e., the relative velocity between two inertial systems is the same and the \( y \) and \( z \) coordinates transform in a Galilean manner. Some researchers and teachers believe that the arguments for the two assumptions are questionable, even if they may seem to be common sense. The invariance of the relative velocity between two inertial coordinates is an obvious case. Because neither the unit length nor the unit time is directly comparable in two systems, the relative velocities of the two systems might not be the same [9].

In the following, we derive the LT with the fewest assumptions or postulates possible. Indeed, we use fundamental postulates to exactly prove the necessary assumptions. First, the invariance of the relative velocity between two inertial coordinates is exactly proven using the constancy of the speed of light and a thought experiment. Second, on the basis of the above straightforward method and results, length contraction...
and time dilation are directly derived. Finally, the LT is exactly formulated without uncalled-for assumptions. Our derivation of the LT is perhaps one of the most perfect derivations.

2. Proof of the invariance of relative velocity between two inertial frames

We suppose two inertial reference frames $K$ and $K'$. The coordinate axes of the two frames are parallel. $K'$ moves with a constant speed $u$ relative to $K$ in the positive direction along the common $x$-$x'$ axis, as viewed from $K$. For simplicity, let the origins $O$ and $O'$ coincide at the initial time $t = t' = 0$.

To prove the invariance of the relative velocity between two inertial frames, let us consider a thought experiment. If a light source at rest at the origin $O$ in $K$ is flashed on and off rapidly at $t = t' = 0$, the constancy of the speed of light implies that observers in both $K$ and $K'$ will see the wavefront moving outward from the respective origins with speed $c$. When the wavefront arrives at the two points $A$ and $B$ on the $x$-axis at a time $t$ in $K$, points $A'$, $M'$, $C'$, and $B'$ on the $x'$-axis coincide with points $A$, $O$, $C$, and $B$ on the $x$-axis, respectively, as seen from $K$. The above is shown in Figure 1, in which $K'$ is at rest relative to $K$.

![Figure 1: In K, the wavefront arrives at two points A and B on the x-axis at time t.](image)

Relations in Figure 1, we obtain

$$\overline{AB} = 2\overline{AO} = 2\overline{OB} = 2ct,$$  \hspace{1cm} (1)

$$\overline{OC} = ut = u\overline{AB}/2c.$$ \hspace{1cm} (2)

where $c$ is the velocity of light in a vacuum.

Suppose $\overline{A'B'}$, $\overline{A'M'}$, $\overline{M'B'}$, and $\overline{M'O'}$ are the distances between the respective two points in $K'$. In Figure 1, they must be the proper length in $K'$ because observers are at rest in $K$. Therefore, their moving lengths, as measured in $K$ at time $t$, are respectively $\overline{AB}$, $\overline{AO}$, $\overline{OB}$, and $\overline{OC}$. Next, we let $\overline{OB} = a\overline{M'B'}$, where $a$ is an arbitrary factor. Because the points $A'$ and $M'$ are moving with constant speed $u$ relative to $K$, $M'$ must move from $O$ to $B$ when $A'$ moves from $A$ to $O$. Thus, the moving length of $\overline{A'M'}$ is also $\overline{OB}$. Because the moving length measured does not depend on time, we still have $\overline{OB} = a\overline{M'B'}$. Thus, $\overline{AO} = a\overline{A'M'}$ and $\overline{AB} = a\overline{A'B'}$. Therefore, the arbitrary factor $a$ is independent of the proper length; that is, proportional relations between any proper length in $K'$ and the moving length in $K$ must have a common linearity proportional factor $a$. This is basically an essential conclusion of the uniform linear motion. Thus, we have

$$\overline{AO} = \alpha \overline{A'M'}, \overline{OB} = \alpha \overline{M'B'}, \overline{AC} = \alpha \overline{A'O'}, \overline{CB} = \alpha \overline{O'B'}, \overline{OC} = \alpha \overline{M'O'}, \overline{AB} = \alpha \overline{A'B'}.$$ \hspace{1cm} (3)

Next, let us change perspectives. Observers at rest in $K'$ see that the wavefront is emitted from the origin $O'$ at $t' = 0$ and moves to $B'$ at time $t'(B')$. Similarly, the wavefront arrives at $A'$ at time $t'(A')$. According to the constancy of the speed of light and equations (1), (2), and (3), we can calculate $t'(B')$ and $t'(A')$:

$$t'(B') = \overline{O'B'}/c = \frac{(c - u)t}{\alpha c},$$  \hspace{1cm} (4)

$$t'(A') = \overline{A'O'}/c = \frac{(c + u)t}{\alpha c}.$$ \hspace{1cm} (5)

Because $\overline{A'O'}$ is greater than $\overline{O'B'}$, $t'(A')$ is greater than $t'(B')$. This is also the famous relativity of simultaneity. Combining the above equations, we have

$$t'(A') - t'(B') = \frac{2ut}{\alpha c} = \frac{u}{c^2} \overline{A'B'}.$$ \hspace{1cm} (6)
Expression (6) allows us to draw an important conclusion; the physical meaning of the expression is that we can calculate the time interval between any two events in $K'$ when the two events occur on the two points in $K$ simultaneously. In Figure 1, because $M'$ coincides with $O$ and $B'$ coincides with $B$ at time $t$ simultaneously in $K$, the time interval between the two points $M'$ and $B'$ is $t’ (M’) – t’ (B’)$, as measured in $K'$. In the same way, we have $t’ (O’) – t’ (B’)$.

From expression (6), we obtain

$$t’ (M’) – t’ (B’) = \frac{u}{c^2} \frac{OB}{\alpha} = \frac{ut}{\alpha c}.$$  \hspace{1cm} (7)

$$t’ (O’) – t’ (B’) = \frac{u}{c^2} \frac{OB}{\alpha} = \frac{(c – u)ut}{\alpha c^2}.$$  \hspace{1cm} (8)

Solving the above equations for $t’ (M’)$ and $t’ (O’)$, we have

$$t’ (M’) = \frac{t}{\alpha}, t’ (O’) = \frac{1 – u^2/c^2}{\alpha} t.$$  \hspace{1cm} (9)

Now let us calculate the velocity $u’$ at which $K$ moves relative to $K’$ in the negative direction along the common $x-x’$ axis, as measured in $K’$. Because every point in $K$ moves with constant speed $u’$ relative to $K’$ in the negative $x’$-direction, we only calculate the speed at which the origin $O$ moves. The observers at rest in $K’$ see that $O$ coincides with $O'$ when the flash is emitted from $O’$ at $t’ = 0$ and $O$ coincides with $M’$ at $t’ (M’)$ in $K’$. Thus, the moving time interval of the origin $O$ from $O’$ to $M’$ is $t’ (M’)$, and the moving distance of origin $O$ is $OM’$, as measured in $K’$. Therefore, the moving velocity of the origin $O$ relative to $K’$ must be $u’ = OM’/ t’ (M’)$.

Combining equations (2), (3) and (9), we find

$$u’ = \frac{OM’}{t’ (M’)} = \frac{ut \alpha}{\alpha t} = u.$$  \hspace{1cm} (10)

We have thus proved the conclusion that the relative velocity between two inertial frames is invariant.

3. Derivation of length contraction and time dilation

To resolve the common proportional factor $\alpha$, we consider that observers at rest in $K’$ see the wavefront arriving at $B’$ at time $t’ (B’)$, as shown in Figure 2. Our question is how to determine the position of $A$ in $K’$ at time $t’ (B’)$. Because $A$ and $A’$ coincide at time $t’ = t’ (A)$ in $K’$, $A$ and a point $E’$ between the two points $A’$ and $B’$ coincide at time $t’ = t’ (B’)$. In addition, the distance between $A’$ and $E’$ is $A’E’ = u [t’ (A’) – t’ (B’)]$.

$$\text{As seen from } K’ \text{ at time } t’ (B’).$$

From expression (6), we obtain

$$\overrightarrow{A’E’} = \frac{u^2}{c^2} \overrightarrow{A’B’}.$$  \hspace{1cm} (11)

In Figure 2, because the observers at rest in $K’$ measure the length between the moving points $A$ and $B$ at the same time, the distance between $A$ and $B$ is the proper length in $K$. Thus, $\overrightarrow{EB’}$ is moving length of $\overrightarrow{AB}$, as seen in $K’$. Because $u’ = u$, proportional factor between the moving and proper lengths, as measured in $K’$, is the same as the above $\alpha$ factor, as measured in $K$. Therefore, we have $\overrightarrow{EB’} = \alpha \overrightarrow{AB}$. According to the geometrical relations in Figure 2, we have $\overrightarrow{A’E’} + \overrightarrow{EB’} = \overrightarrow{A’B’}$. Combining the above equations, we obtain

$$\frac{u^2}{c^2} \overrightarrow{A’B’} + \alpha \overrightarrow{AB} = \overrightarrow{A’B’}.$$  \hspace{1cm} (12)

Because the scale factor $\alpha$ cannot be negative, solving equation (12) for $\alpha$, we find

$$\alpha = \sqrt{1 – u^2/c^2}.$$  \hspace{1cm} (13)

Ultimately, we have proven the length contraction effect; i.e., $\overrightarrow{AB} = \alpha \overrightarrow{A’B’}$. Next we express the time dilation effect on the basis of equations (9) and (13); that is, $t’ (O’) = at$. Therefore, the length contraction effect and time dilation effect have been derived.
4. Derivation of the Lorentz transformation

To derive the Lorentz coordinate transformation, we refer to Figure 3, which initial conditions are the same as Figure 1. For simplicity, we first consider the one-dimensional case. We suppose the time and space coordinates of an arbitrary point on the common \( x-x' \) axis are \( P(x, t) \) and \( P'(x', t') \) in \( K \) and \( K' \), respectively. Thus, \( P(x, t) \) and \( P'(x', t') \) coincide at time \( t \), and the distance from \( O \) to \( N \) at time \( t \) is \( ut \), as seen from \( K \).

The coordinate \( x' \) from \( O' \) to \( P' \) is a proper length in \( K' \); thus, in \( K \), the distance from \( N \) to \( P \) is \( \alpha x' \). Therefore, the distance from \( O \) to \( P \) is \( x \):

\[
x = ut + \alpha x'.
\]

Solving the equation for \( x' \), we obtain the first equation of the Lorentz coordinate transformation:

\[
x' = \frac{x - ut}{\alpha}.
\]  

(15)

Because \( O' \) and \( P' \) coincide respectively with \( N \) and \( P \) at time \( t \), as seen from \( K \), the difference between time \( t'(O') \) and time \( t'(P') = t' \) follows equation (6); i.e.,

\[
t'(O') - t' = \frac{u}{c^2} O'P' = \frac{u}{c^2} x'.
\]

(16)

Now we consider how to derive the transformation relations between \( y \) and \( y' \) and between \( z \) and \( z' \) from the constancy of the speed of light. Let us consider another thought experiment. As we discussed previously for Figure 1, we suppose that a wavefront is emitted from \( O' \) at \( t' = 0 \) and moves in the positive direction along the \( y' \) axis, as viewed from \( K' \). After the wavefront travels a distance \( y' \), it is reflected by a mirror and returns to \( O' \) at time \( t' \). Thus, the wavefront moves a total distance \( 2y' \) and the interval time is \( t' \). Therefore, according to the constancy of the speed of light, we find

\[
y' = \frac{ct'}{2}.
\]

(18)

The same propagation of the wavefront may also be considered relative to \( K \), in which case the constancy of the speed of light must also be satisfied. The round-trip time measured in \( K \) is a different interval \( t \). In \( K \), the wavefront leaves and returns at different points on the \( x \) axis. During the time \( t \), the origin \( O' \) moves a distance \( ut \) relative to \( K \). The distance between the mirror and origin \( O \) is \( y \) when the two origins \( O' \) and \( O \) coincide. Therefore, according to the constancy of the speed of light, we have

\[
\sqrt{y^2 + (ut/2)^2} = ct/2.
\]

(19)

Substituting equation (18) and \( t' = at \) into equation (19), we find

\[
y' = y.
\]

(20)

In the same way, we obtain the transformation relation between \( z \) and \( z' \):

\[
z' = z.
\]

(21)

We have thus exactly derived the LT. Equations (15), (17), (20), and (21) are the Lorentz coordinate direct transformation. The so-called Lorentz coordinates inverse transformation can be found from the direct LT by direct calculation.

5. Conclusion

The theory of relativity is the greatest theory in physics, and at its heart is the LT. However, the theory of relativity is sometimes criticized. One doubt is that
too many assumptions are made, especially in the derivation of the LT. To address these concerns, researchers have studied the derivation of the LT without using uncalled-for assumptions. However, none succeeded in completing it. We have thus derived the LT exactly.

It should be noted that we have not made any assumption except for the constancy of the speed of light and uniform linear motion in the above derivation of the LT. Yet, our derivation is simple and clear and is perhaps the most perfection. Thus, our systematic method will be important in the study, application, and teaching of the special theory of relativity.

The author expresses his gratitude to Prof. Wang Qiang for useful discussions.

References