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Recurrence triangle for Adomian polynomials

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ABSTRACT

In this paper a recurrence technique for calculating Adomian polynomials is proposed, the convergence of the series for the Adomian polynomials is discussed, and the dependence of the convergent domain of the solution's decomposition series $\sum_{n=0}^{\infty} u_n$ on the initial component function u_0 is illustrated. By introducing the index vectors of the Adomian polynomials the recurrence relations of the index vectors are discovered and the recurrence triangle is given. The method simplifies the computation of the Adomian polynomials. In order to obtain a solution's decomposition series with larger domain of convergence, we illustrate by examples that the domain of convergence can be changed by choosing a different u_0 and a modified iteration.

1. Introduction

The Adomian decomposition method [1–3] has been used to give analytic approximation for a large class of linear and nonlinear functional equations, including differential equations, integral equations, integro-differential equations, etc.

Let us recall the basic principles of this technique by a second order ordinary differential equation in the form

$$Lu + Ru + f(u) = g(t), \tag{1}$$

where $L = \frac{d^2}{dt^2}$, R is the remaining linear operator grouping the lower order derivatives, f represents an analytic nonlinear operator and g is a given function.

Integrating (1) yields

$$u = u(0) + u'(0)t + L^{-1}g - L^{-1}Ru - L^{-1}f(u), \tag{2}$$

for the initial value problems, where L^{-1} is the two-fold definite integration operator from 0 to t. For boundary value problems indefinite integrations are used and constants are evaluated from the given conditions.

The decomposition method consists in looking for the solution in the series form $u = \sum_{n=0}^{\infty} u_n$. The nonlinear operator is decomposed as

$$f(u) = \sum_{n=0}^{\infty} A_n, \tag{3}$$

where A_n depends on $u_0, u_1, \ldots, u_n$, called the Adomian polynomials that are obtained by writing

$$u(\lambda) = \sum_{n=0}^{\infty} u_n \lambda^n, \quad f(u(\lambda)) = \sum_{n=0}^{\infty} A_n \lambda^n, \tag{4}$$

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where λ is a parameter. From (4) the A_n's are deduced

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(\sum_{n=0}^{\infty} u_n\lambda^n\right)\right]_{\lambda=0}, \quad n = 0,1,2,\ldots. \tag{5}$$

The first few Adomian polynomials are

$$\begin{aligned} A_0 &= f(u_0), \\ A_1 &= f'(u_0)u_1, \\ A_2 &= f'(u_0)u_2 + f''(u_0)\frac{u_1^2}{2!}, \\ A_3 &= f'(u_0)u_3 + f''(u_0)u_1u_2 + f^{(3)}(u_0)\frac{u_1^3}{3!}, \\ &\ldots. \end{aligned} \tag{6}$$

The decomposition method consists in identifying the u_n's by means of the formulae

$$u_0 = u(0) + u'(0)t + L^{-1}g, \tag{7}$$

$$u_n = -L^{-1}Ru_{n-1} - L^{-1}A_{n-1}, \quad n = 1,2,\ldots. \tag{8}$$

Convergence of this method was studied in [4–8]. Especially if in (2) the operator $L^{-1}R + L^{-1}f$ is contracting in the suitable Banach space the scheme (6)–(8) gives a convergent series $\sum u_n$ the sum of which is the unique solution of (1) [4]. For physical system the solution is assumed to be existing [6]. The m-term approximation $\phi_m = \sum_{i=0}^{m-1} u_i$ serves as a practical solution.

The key of the method is to decompose the nonlinear term in the equation into a series of polynomials A_n. How to construct a practical technique for calculating the Adomian polynomials A_n has been attracting much attention and a lot of contribution has been made [7–22].

The earlier calculations utilize (5) and the equality $\frac{d^n}{d\lambda^n}f(u(\lambda)) = \sum_{v=1}^{n} f^{(v)}(u(\lambda))c_\lambda(v,n)$ with the recurrence rule for $c_\lambda(v,n)$ [2,9]

$$c_\lambda(i,j) = \frac{dc_\lambda(i,j-1)}{d\lambda} + \frac{du(\lambda)}{d\lambda}c_\lambda(i-1,j-1), \quad 1 \leqslant i \leqslant j,$$

letting $c_\lambda(0,0) = 1,\ c_\lambda(0,j) = 0(j \geqslant 1),\ c_\lambda(i,j) = 0(i > j)$. Rach's Rule [1,3,10] $A_n = \sum_{v=1}^{n} f^{(v)}(u_0)C(v,n),\ n > 0$, simplifies the computation, where $C(v,n)$ are products (or sums of products) of v components of u whose subscripts sum to n, divided by the factorial of the number of repeated subscripts.

Abbaoui and Cherruault [7] gave a formula for A_n by dividing n into all possible decreasing sequences of nonnegative integers. Biazar et al [12] and Zhu et al [13] used parametrization (4) to get A_n. Wazwaz [14] and Abdelwahid [15], without parametrization, obtained A_n by expanding $f\left(\sum_{i=0}^{\infty} u_i\right)$ and then regrouping such that the sum of the subscripts of the components of u in each terms is the same. Babolian and Javadi [16] gave a special operator to derive A_n recursively (see also Gu and Li [17]). Biazar and Shafiof [18] gave a recurrence algorithm through the parameter and derivatives. Azreg-Aïnou [19] studied the properties of the Adomian polynomials through the system of equations. In [8] Rach defined truncating operator of Taylor expansion to obtain the A_n. Symbolic implementation of the algorithms by using software Mathematica or Maple was considered in [17,19–22].

In the next section we introduce the index vectors for the Adomian polynomials, and discover the recurrence relations of the index vectors. Thus a new and simple algorithm for the Adomian polynomials A_n is obtained. We discuss the convergence of the Adomian polynomial series and the solution's decomposition series in Section 3.

We note that the m-term approximation ϕ_m requires all the $A_0, A_1, \ldots, A_{m-2}$. So the recursive computation of the A_n's is more advisable in order to reduce the volume of calculations. The algorithms in [9,16–18] are recursive, involving parametrization and derivatives, which are unnecessary in the present algorithm.

2. Recurrence technique for index vectors

We begin with the following expression for A_n (Rach's Rule) [1,3,10]

$$A_n = \sum_{k=1}^{n} f^{(k)}(u_0)C(k,n), \tag{9}$$

where $C(k,n)$ is homogeneous polynomial of degree k in $u_1,\ldots,u_n$, we write it out explicitly

$$C(k,n) = \sum_{p_1+2p_2+\cdots+np_n=n,\ p_1+p_2+\cdots+p_n=k} \frac{u_1^{p_1}u_2^{p_2}\cdots u_n^{p_n}}{p_1!p_2!\cdots p_n!}. \tag{10}$$

Each summand term of A_n corresponds to an n-dimensional vector $(p_1,p_2,\ldots,p_n)$ with nonnegative integer entries. We call $(p_1,p_2,\ldots,p_n)$ to be an index vector of A_n.

For given positive integers $n,\ k\ (k \leqslant n)$, let S_n^k denote the set of all nonnegative integer solution vectors $(x_1, x_2, \ldots, x_n)$ of system of indeterminate equations

$$x_1 + 2x_2 + \cdots + nx_n = n, \quad x_1 + x_2 + \cdots + x_n = k. \tag{11}$$

Then $\bigcup_{k=1}^n S_n^k$ is the set of all index vectors of A_n. If $S_n^1,\ S_n^2, \ldots,\ S_n^n$ are obtained then A_n can be get

$$A_n = \sum_{k=1}^{n} \sum_{(p_1,\ldots,p_n) \in S_n^k} f^{(p_1+p_2+\cdots+p_n)}(u_0) \frac{u_1^{p_1} u_2^{p_2} \cdots u_n^{p_n}}{p_1! p_2! \cdots p_n!}. \tag{12}$$

Each vector in S_n^k n-dimensional, and its dimension will not be indicated if no confusion is caused.

The following Lemmas are easy to be verified.

Lemma 1. *S_n^1 contains one vector as follows: $S_1^1 = \{(1)\},\ S_n^1 = \{(0, \ldots, 0, 1)\}$.*

Lemma 2. *For each vector $(p_1, \ldots, p_n) \in S_n^k (2 \leqslant k \leqslant n)$, the last $k-1$ entries are zero, that is $p_{n-k+2} = \cdots = p_n = 0$.*

The recurrence relations of the index vectors are given as follows.

Theorem 1. *For a given $n > 1$, if $2 \leqslant k \leqslant \left[\frac{n}{2}\right]$ (this happens only if $n \geqslant 4$), then*

$$S_n^k = \left\{(p_1+1, p_2, \ldots, p_{n-1}, 0) | (p_1, p_2, \ldots, p_{n-1}) \in S_{n-1}^{k-1}\right\} \bigcup \left\{(0, p_1, p_2, \ldots, p_{n-k}, 0, \ldots, 0) | (p_1, p_2, \ldots, p_{n-k}) \in S_{n-k}^k\right\}, \tag{13}$$

if $\left[\frac{n}{2}\right] < k \leqslant n$, then

$$S_n^k = \left\{(p_1+1, p_2, \ldots, p_{n-1}, 0) | (p_1, p_2, \ldots, p_{n-1}) \in S_{n-1}^{k-1}\right\}. \tag{14}$$

Proof. First, split S_n^k into two parts $S_n^k = S_{n,1}^k \bigcup S_{n,0}^k$, where $S_{n,1}^k$ is composed of vectors in S_n^k with non-zero first entry, while in $S_{n,0}^k$ the first entry of a vector is 0. We will make certain $S_{n,1}^k$ and $S_{n,0}^0$, respectively.

For any $(p_1, p_2, \ldots, p_{n-1}) \in S_{n-1}^{k-1}$, it satisfies

$$p_1 + 2p_2 + \cdots + (n-1)p_{n-1} = n-1, \quad p_1 + p_2 + \cdots + p_{n-1} = k-1.$$

Let $p_n = 0$, the above can be written as

$$(p_1+1) + 2p_2 + \cdots + np_n = n, \quad (p_1+1) + p_2 + \cdots + p_n = k.$$

This means $(p_1+1, p_2, \ldots, p_{n-1}, 0) \in S_{n,1}^k$. Conversely, for any $(p_1, p_2, \ldots, p_n) \in S_{n,1}^k,\ p_n = 0$ by Lemma 2, accordingly

$$p_1 + 2p_2 + \cdots + (n-1)p_{n-1} = n, \quad p_1 + p_2 + \cdots + p_{n-1} = k.$$

It follows that $(p_1 - 1, p_2, \ldots, p_{n-1}) \in S_{n-1}^{k-1}$ by subtracting 1 on both sides. Consequently, $S_{n,1}^k$ is determined

$$S_{n,1}^k = \left\{(p_1+1, p_2, \ldots, p_{n-1}, 0) | (p_1, p_2, \ldots, p_{n-1}) \in S_{n-1}^{k-1}\right\}. \tag{15}$$

Suppose $S_{n,0}^k \neq ø$ and $(0, p_2, \ldots, p_n) \in S_{n,0}^k$. So we have $2p_2 + \cdots + np_n = n,\ p_2 + \cdots + p_n = k$. It follows that $2k \leqslant n$. Thus $k \leqslant \left[\frac{n}{2}\right]$.

Therefore, if $\left[\frac{n}{2}\right] < k \leqslant n$, then $S_{n,0}^k = ø$, and this implies $S_n^k = S_{n,1}^k$. (14) is proved in light of (15).

If $2 \leqslant k \leqslant \left[\frac{n}{2}\right]$ (in this case $n \geqslant 4$), $S_{n,0}^k$ needs to be identified. For any $(p_1, p_2, \ldots, P_{n-k}) \in S_{n-k}^k$, it satisfies

$$p_1 + 2p_2 + \cdots + (n-k)p_{n-k} = n-k, \quad p_1 + p_2 + \cdots + p_{n-k} = k.$$

Adding the two equations yields $2p_1 + 3p_2 + \cdots + (n-k+1)p_{n-k} = n$. This implies $(0, p_1, p_2, \ldots, P_{n-k}, 0, \ldots, 0) \in S_{n,0}^k$. Conversely, for any $(0, p_2, \ldots, p_n) \in S_{n,0}^k$. It follows from Lemma 2 that $p_{n-k+2} = \cdots = p_n = 0$, moreover

$$2p_2 + 3p_3 + \cdots + (n-k+1)p_{n-k+1} = n, \quad p_2 + p_3 + \cdots + p_{n-k+1} = k.$$

The difference of them implies $(p_2, p_3, \ldots, p_{n-k+1}) \in S_{n-k}^k$ Therefore, it is derived that

$$S_{n,0}^k = \{(0, p_1, p_2, \ldots, p_{n-k}, 0, \ldots, 0) | (p_1, p_2, \cdots, p_{n-k}) \in S_{n-k}^k\}. \tag{16}$$

By using (15) and (16), (13) is proved. The proof of the Theorem is completed. □

Based on Lemmas 1 and 2 and Theorem 1 the recurrence triangle of index vectors for the Adomian polynomials can be listed easily as Table 1. In the position (n,k) we list the vectors in S_n^k. The column $k = 1$ is given by Lemma 1. We list S_2^2, S_3^2, S_3^3 by (14), S_4^2 by (13), S_4^3, S_4^4 by (14), S_5^2 by (13), S_5^3, S_5^4, S_5^5 by (14), S_6^2, S_6^3 by (13), S_6^4, S_6^5, S_6^6 by (14), and so on. For a vector in $S_{n,0}^k$ in (16) it is denoted by italic entries for emphasis. As space is limited we list till $n = 6$, juxtapose the entries of vectors and omit the last k-1 zero entries in last three columns.

The algorithm in Theorem 1 is convenient and easy for both hand calculations and computer programs.

Table 1
Recurrence triangle of index vectors for Adomian polynomials.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=1$	(1)					
$n=2$	(01)	(20)				
$n=3$	(001)	(110)	(300)			
$n=4$	(0001)	(1010) *(0200)*	(2100)	(4..)		
$n=5$	(00001)	(10010) *(01100)*	(20100) (12000)	(31..)	(5..)	
$n=6$	(000001)	(100010) *(010100)* *(002000)*	(200100) (111000) *(030000)*	(301..) (220..)	(41..)	(6..)

Having the $S_n^k(k=1,2,\ldots,n)$ in hands one can write out A_n. For example, from the rows $n=4,5,6$ we get

$$A_4 = f'(u_0)u_4 + f''(u_0)\left(u_1u_3 + \frac{u_2^2}{2!}\right) + f^{(3)}(u_0)\frac{u_1^2u_2}{2!} + f^{(4)}(u_0)\frac{u_1^4}{4!},$$

$$A_5 = f'(u_0)u_5 + f''(u_0)(u_1u_4 + u_2u_3) + f^{(3)}(u_0)\left(\frac{u_1^2u_3}{2!} + \frac{u_1u_2^2}{2!}\right) + f^{(4)}(u_0)\frac{u_1^3u_2}{3!} + f^{(5)}(u_0)\frac{u_1^5}{5!},$$

$$A_6 = f'(u_0)u_6 + f''(u_0)\left(u_1u_5 + u_2u_4 + \frac{u_3^2}{2!}\right) + f^{(3)}(u_0)\left(\frac{u_1^2u_4}{2!} + u_1u_2u_3 + \frac{u_2^3}{3!}\right)$$
$$+ f^{(4)}(u_0)\left(\frac{u_1^3u_3}{3!} + \frac{u_1^2u_2^2}{2!2!}\right) + f^{(5)}(u_0)\frac{u_1^4u_2}{4!} + f^{(6)}(u_0)\frac{u_1^6}{6!}.$$

From Theorem 1 the recursive algorithms of $C(k,n)$ in (9) are as follows.

Corollary 1. *For any $n \geqslant 1$,*

$$C(1,n) = u_n. \tag{17}$$

As $n \geqslant 2$ and $\left[\frac{n}{2}\right] < k \leqslant n$,

$$C(k,n) = C(k-1,n-1)|_{p1\to p1+1}. \tag{18}$$

As $n \geqslant 4$ and $2 \leqslant k \leqslant \left[\frac{n}{2}\right]$,

$$C(k,n) = C(k-1,n-1)|_{p_1\to p_1+1} + C(k,n-k)|_{u_j\to u_{j+1}}. \tag{19}$$

In the Corollary $p_1 \to p_1+1$ stands for replacing $\frac{u_1^{p_1}}{p_1!}$ by $\frac{u_1^{p_1+1}}{(p_1+1)!}$, where $p_1 \geqslant 0$.

For the nonlinear function $f(u)=u^m$, where m is a positive integer more than 1, the Adomian polynomial A_n has special form if $n > m$, i.e.

$$A_n = \sum_{k=1}^{m} f^{(k)}(u_0)C(k,n). \tag{20}$$

Especially, for the function $f(u)=u^2$ the Adomian polynomials are:

$$A_0 = u_0^2,\ \ A_1 = 2u_0u_1,\ \ A_2 = 2u_0u_2 + u_1^2,\ \ A_3 = 2u_0u_3 + 2u_1u_2, \ldots,\ \ A_n = \sum_{i=0}^{n} u_iu_{n-i}. \tag{21}$$

3. Discussions for convergence

Compared with the Picard iterative scheme the Adomian decomposition method decomposes the solution into a sequence of "relatively easier" equations, the difficulty has been transferred to the decomposition of the nonlinear term $f(u)$ [23,24].

The Adomian polynomials are requisite for Wazwaz's new modification of the Adomian decomposition method [25] and Adomian's modified decomposition (i.e. power series solution) [1]. In [26] Adomian polynomials are applied to the variational iteration method to solve the nonlinear physical models.

On the convergence of the Adomian polynomial series, it is already realized that the series $\sum_{n=0}^{\infty} A_n$ is a rearrangement of the Taylor expansion of $f\left(\sum_{n=0}^{\infty} u_n\right)$ about the solution's initial component function u_0 [6,8]. We show this explicitly as follows.

Suppose $f(u)$ to be analytic and $\sum_{n=0}^{\infty} u_n$ absolutely convergent. The Taylor expansion of $f\left(\sum_{n=0}^{\infty} u_n\right)$ about u_0 reads

$$f\left(\sum_{n=0}^{\infty} u_n\right) = f(u_0) + \sum_{k=1}^{\infty} f^{(k)}(u_0) \frac{\left(\sum_{i=1}^{\infty} u_i\right)^k}{k!}. \tag{22}$$

Using the multinomial expansion and then regrouping yield

$$\frac{\left(\sum_{i=1}^{\infty} u_i\right)^k}{k!} = \sum_{\sum_{j=1}^{\infty} p_j = k} \frac{u_1^{p1} u_2^{p2} \cdots}{p_1! p_2! \cdots} = \sum_{n=k}^{\infty} \sum_{\sum_{j=1}^{\infty} p_j = k,\ \sum_{j=1}^{\infty} j p_j = n} \frac{u_1^{p1} u_2^{p2} \cdots}{p_1! p_2! \cdots}.$$

Inserting it to (23) and then exchanging the first two sums, and observing that the system of equations $\sum_{j=1}^{\infty} p_j = k,\ \sum_{j=1}^{\infty} j p_j = n$ is equivalent to $\sum_{j=1}^{n} p_j = k,\ \sum_{j=1}^{n} j p_j = n$, and $p_j = 0$ for $j > n$ result in

$$f\left(\sum_{n=0}^{\infty} u_n\right) = f(u_0) + \sum_{n=1}^{\infty} \sum_{k=1}^{n} f^{(k)}(u_0) \sum_{\sum_{j=1}^{n} p_j = k,\ \sum_{j=1}^{n} j p_j = n} \frac{u_1^{p_1} u_2^{p_2} \cdots u_n^{p_n}}{p_1! p_2! \cdots p_n!} = f(u_0) + \sum_{n=1}^{\infty} A_n.$$

The absolute convergence guarantees the sum is invariant.

Now we illustrate by examples that the limitation, and the dependence on the initial component function u_0, of the convergent domain of the decomposition series $\sum_{n=0}^{\infty} u_n$.

Example 1. Consider the nonlinear ordinary differential equation

$$u_t + e^u = 0, \quad u(0) = 1. \tag{23}$$

The solution of the equation is

$$u(t) = 1 - \ln(1 + et), \quad t > -e^{-1}.$$

Let $L_t = \frac{d}{dt},\ L_t^{-1} = \int_0^t (\cdot) dt$, the Eq. (23) is written as

$$u = 1 - L_t^{-1} e^u. \tag{24}$$

Let $u = \sum_{n=0}^{\infty} u_n$. The Adomian polynomials for $f(u) = e^u$ are

$$A_0 = e^{u_0},\ A_1 = e^{u_0} u_1,\ A_2 = e^{u_0}\left(u_2 + u_1^2/2\right), \ldots,\ A_n = e^{u_0} \sum_{k=1}^{n} C(k, n).$$

By the scheme (7) and (8) we have

$$u_0 = 1,\ u_n = -L_t^{-1} A_{n-1}, \quad n = 1, 2, \ldots.$$

Calculating $A_0, u_1, A_1, u_2, \ldots$ in tern yields

$$u_1 = -et,\ u_2 = \frac{(et)^2}{2},\ u_3 = -\frac{(et)^3}{3},\ u_4 = \frac{(et)^4}{4}, \ldots.$$

Thus the decomposition series is $u = 1 - et + \frac{(et)^2}{2} - \frac{(et)^3}{3} + \cdots$ with the domain of convergence $-e^{-1} < t \leqslant e^{-1}$ and the radius of convergence e^{-1}.

In order to obtain a solution's decomposition series with larger domain of convergence we try to change the choice of u_0 since all the u_n's are derived from u_0.

We take arbitrarily $u_0 < 1 + \ln 2$. Let $u_0 = 1 - \ln(1 + et_0),\ t_0 > -\frac{1}{2e}$. Write the Eq. (24) as

$$u = 1 - \ln(1 + et_0) + \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{et_0}{1 + et_0}\right)^n - L_t^{-1} e^u.$$

Applying the iterative scheme

$$u_0 = 1 - \ln(1 + et_0),\ u_n = \frac{1}{n} \left(\frac{et_0}{1 + et_0}\right)^n - L_t^{-1} A_{n-1}, \quad n = 1, 2, \ldots,$$

leads to

$$A_0 = \frac{e}{1 + et_0}, \quad u_1 = -\frac{e(t - t_0)}{1 + et_0},$$

$$A_1 = -\left(\frac{e}{1 + et_0}\right)^2 (t - t_0), \quad u_2 = \frac{1}{2} \left(\frac{e(t - t_0)}{1 + et_0}\right)^2,$$

$$A_2 = \left(\frac{e}{1 + et_0}\right)^3 (t - t_0)^2, \quad u_3 = -\frac{1}{3} \left(\frac{e(t - t_0)}{1 + et_0}\right)^3, \ldots.$$

In this case the decomposition series $u=\sum_{n=0}^{\infty}u_n$ bears the domain of convergence $-e^{-1}<t\leqslant 2t_0+e^{-1}$ and the radius of convergence t_0+e^{-1}. The radius of convergence is a function of u_0 as $\rho=e^{-u_0}$.

Example 2. Consider the first-order nonlinear partial differential equation

$$u_t+uu_x=0,\quad u(x,0)=x. \tag{25}$$

This equation has the solution $u(x,t)=\frac{x}{1+t},\ t>-1$.

Let $L_t=\frac{\partial}{\partial t},\ L_x=\frac{\partial}{\partial x}$. Applying the integration operator L_t^{-1} to both sides yields

$$u=x-\frac{1}{2}L_t^{-1}L_x(u^2). \tag{26}$$

Decomposing u and using (21), applying the scheme (6)–(8) one obtains

$$u_0=x,\ u_1=-\frac{1}{2}L_t^{-1}L_xA_0=-xt,\ u_2=xt^2,\ u_3=-xt^3,\ u_4=xt^4,\ldots.$$

The decomposition series $u=x-xt+xt^2-xt^3+\cdots$ converges for $|t|<1$.

Now we consider the effect of different choice for u_0 on the domain of convergence.

We take $u_0=\frac{x}{1+t_0}$, where $t_0>-\frac{1}{2}$. Adapt the Eq. (26) as

$$u=u_0+\frac{x}{1+t_0}\sum_{n=1}^{\infty}\left(\frac{t_0}{1+t_0}\right)^n-\frac{1}{2}L_t^{-1}L_x(u^2).$$

Applying the scheme

$$u_0=\frac{x}{1+t_0},\ u_n=\frac{x}{1+t_0}\left(\frac{t_0}{1+t_0}\right)^n-\frac{1}{2}L_t^{-1}L_xA_{n-1},\quad n=1,2,\ldots,$$

yields

$$u_1=-\frac{x(t-t_0)}{(1+t_0)^2},\ u_2=\frac{x(t-t_0)^2}{(1+t_0)^3},\ u_3=-\frac{x(t-t_0)^3}{(1+t_0)^4},\ldots.$$

Now the decomposition series $u=\sum_{n=0}^{\infty}u_n$ converges for $|t-t_0|<1+t_0$. The radius of convergence depends on $u_0:\rho=\frac{x}{u_0}$.

The examples indicate we can improve the domain of convergence by a suitable choice u_0.

In following example the decomposition series contains the fractional power.

Example 3. Let us consider the linear ($m=1$) and nonlinear ($m=2$) integral equations with the weakly singular kernel

$$u(t)=1+\int_0^t\frac{(t-s)^{\mu-1}}{\Gamma(\mu)}u^m(s)ds,\quad t\geqslant 0,$$

where $0.5\leqslant\mu\leqslant 1,\ \Gamma$ is the Gamma function.

The equation is equivalent to a fractional differential equation [27]. Denote $Lu(t)=\int_0^t\frac{(t-s)^{\mu-1}}{\Gamma(\mu)}u(s)ds$.

Linear case ($m=1$): If $\mu=1$ the integral equation has the solution $u(t)=e^t$.

If $0.5\leqslant\mu\leqslant 1$ we have successively $u_0=1,\ u_1=Lu_0=\frac{t^\mu}{\Gamma(\mu+1)},\ u_2=Lu_1=\frac{t^{2\mu}}{\Gamma(2\mu+1)},\ldots,\ u_n=Lu_{n-1}=\frac{t^{n\mu}}{\Gamma(n\mu+1)}$. The solution is

$$u(t)=\sum_{k=0}^{\infty}\frac{t^{k\mu}}{\Gamma(k\mu+1)}=E_\mu(t^\mu),\quad t\geqslant 0,$$

where E_μ is the Mittag-Leffler function [27]. As $\mu=1$ the solution degenerates to the exponential function $u(t)=e^t$.

Nonlinear case ($m=2$): In this case we limit $0\leqslant t<1$. If $\mu=1$ the integral equation has the solution $u(t)=1/(1-t)$.

If $0.5\leqslant\mu\leqslant 1$ depending on the Adomian polynomials (21) we have $u_0=1,\ u_1=LA_0=\frac{t^\mu}{\Gamma(\mu+1)},\ u_2=LA_1=\frac{2t^{2\mu}}{\Gamma(2\mu+1)}$, $u_3=\left(4+\frac{\Gamma(2\mu+1)}{\Gamma^2(\mu+1)}\right)\frac{t^{3\mu}}{\Gamma(3\mu+1)},\ u_4=\left(8+\frac{2\Gamma(2\mu+1)}{\Gamma^2(\mu+1)}+\frac{4\Gamma(3\mu+1)}{\Gamma(\mu+1)\Gamma(2\mu+1)}\right)\frac{t^{4\mu}}{\Gamma(4\mu+1)}$.

The five-term approximation of the solution is $\phi_5=\sum_{k=0}^{4}u_k$, which degenerates to $1+t+t^2+t^3+t^4$ as $\mu=1$.

For the applications subsuming boundary value problems in physical equations, even for stochastic systems and modifications for the decomposition methods see [1–3,25,28].

4. Conclusion

The index vectors of the Adomian polynomials are introduced in this article and their recurrence relations are found out. Thus a convenient recurrence algorithm for the Adomian polynomials is obtained. The recurrence operations are simpler than the existing algorithms, do not require parametrization, expanding, derivatives, etc. In addition, we deduce explicitly that the series for the Adomian polynomials $\sum_{n=0}^{\infty}A_n$ is a rearrangement of the Taylor expansion of $f\left(\sum_{n=0}^{\infty}u_n\right)$ about the solution's initial component function u_0. We also indicate by examples the limitation of the domain of convergence of the decomposition series $\sum_{n=0}^{\infty} u_n$, further the domain of convergence can be changed by choosing a different u_0 and a modified iteration. So a suitable u_0 can lead to a larger domain of convergence.

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