Demonstrating Anyonic Fractional Statistics with a Six-Qubit Quantum Simulator

Chao-Yang Lu,[1] Wei-Bo Gao,[1] Otfried Gühne,[2,3] Xiao-Qi Zhou,[1] Zeng-Bing Chen,[1] and Jian-Wei Pan[1,4]

[1]*Hefei National Laboratory for Physical Sciences at Microscale and Department of Modern Physics, University of Science and Technology of China, Hefei, 230026, China*

[2]*Institut für Quantenoptik und Quanteninformation, Österreichische Akademie der Wissenschaften, Technikerstraße 21A, A-6020 Innsbruck, Austria*

[3]*Institut für theoretische Physik, Universität Innsbruck, Technikerstraße 25, A-6020 Innsbruck*

[4]*Physikalisches Institut, Universität Heidelberg, Philosophenweg 12, 69120 Heidelberg, Germany*

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Anyons are exotic quasiparticles living in two dimensions that do not fit into the usual categories of fermions and bosons, but obey a new form of fractional statistics. Following a recent proposal [Phys. Rev. Lett. **98**, 150404 (2007)], we present an experimental demonstration of the fractional statistics of anyons in the Kitaev spin lattice model using a photonic quantum simulator. We dynamically create the ground state and excited states (which are six-qubit graph states) of the Kitaev model Hamiltonian, and implement the anyonic braiding and fusion operations by single-qubit rotations. A phase shift of π related to the anyon braiding is observed, confirming the prediction of the fractional statistics of Abelian $1/2$ anyons.

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Quantum statistics classifies fundamental particles in three dimensions as bosons and fermions. Interestingly, in two dimensions, the laws of physics permit the existence of exotic quasiparticles—anyons—which obey a new statistical behavior, called fractional or braiding statistics [1]. That is, upon exchange of two such particles, the system wave function will acquire a statistical phase which can take any value—hence this name. Anyons have been predicted to live as excitations in fractional quantum Hall (FQH) systems [2–4]. Alternatively, quantum states with anyonic excitations can be artificially designed in spin model systems that possess highly nontrivial ground states with topological order. A prominent example is the Kitaev spin lattice model [5,6], which opened the avenue of fault-tolerant topological quantum computing [7,8].

It is an important goal to manipulate the anyons and demonstrate their exotic statistics. Towards this goal, a number of theoretical schemes have been proposed, both in the FQH regime [9] and in the Kitaev models [10–14]. However, it has proved extremely difficult to experimentally detect the fractional statistics associated with anyon braiding. While recent FQH experiments have revealed some signatures of anyonic statistics [15,16], resolving individual anyons remains elusive [7]. Here, we take a different approach to study the anyonic statistics in the spin models. Following a recent proposal [14], we demonstrate the fractional statistics of anyons by simulation of the Kitaev spin model with a six-photon graph state.

First, let us recall the quantum statistics of elementary particles. A two-particle wave function $\psi(\mathbf{r}_1, \mathbf{r}_2)$ acquires a statistical phase θ upon an adiabatic exchange of two particles, that is, $\psi(\mathbf{r}_2, \mathbf{r}_1) = e^{i\theta}\psi(\mathbf{r}_1, \mathbf{r}_2)$, where $\theta = 0$ for bosons, $\theta = \pi$ for fermions, and θ can be any value ($0 \le \theta \le \pi$) for anyons. It can be seen that a full circulation of a particle around the other one is equivalent to two successive particle exchanges [8]. After such a circulation, both bosons and fermions acquire a trivial phase ($\phi = 2\theta = 0, 2\pi$), but anyons get an observable nontrivial phase ϕ. To detect such phase, we need a system where anyons can be created and braided; the Kitaev model [5] is well suited for this.

The first Kitaev model was designed on a spin lattice with qubits living on the edges [see Fig. 1(a)]. For each vertex v and face f, we consider operators of the form

$$A_v = \prod_{j\in \mathrm{star}(v)} X_j, \qquad B_f = \prod_{j\in \mathrm{boundary}(f)} Z_j, \tag{1}$$

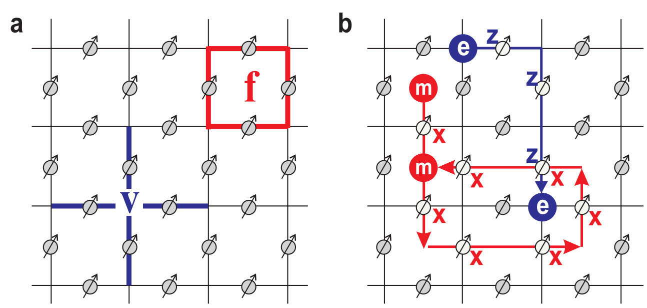

FIG. 1 (color online). (a) The first Kitaev model [5]. The qubits live on the edges, and the stabilizer operators A_v, B_f (1) represent the four-body interactions as illustrated. (b) Creation of quasiparticles and braid of an m particle around an e particle. A pair of e particles (m particles) are created on vertices (faces) by applying a Z (X) operation on the edge qubit. The quasiparticles can be moved horizontally and vertically by repeated applications of Z or X operations. The figure shows an example of how an m particle forms a closed loop around an e particle through a series of moves.

where X (Z) denotes the standard Pauli matrix σ_x (σ_z). These operators A_v, B_f are put together to make up the model Hamiltonian

$$H = -\sum_v A_v - \sum_f B_f. \tag{2}$$

The ground state $|\psi_g\rangle$ of this Hamiltonian (2) is given by $A_v|\psi_g\rangle = |\psi_g\rangle$ and $B_f|\psi_g\rangle = |\psi_g\rangle$ for all vertices and faces. Violations of these conditions cost energy and generate excited states $|\psi_e\rangle$. A quasiparticle is created on the vertex v_i (face f_i) if A_{v_i} (B_{f_i}) acting on the excited state $|\psi_e\rangle$, yields an eigenvalue -1 instead of $+1$ for the ground state. In Refs. [5,14], the quasiparticles on vertices are called "electric charges" (e particles for short) and those on faces are called "magnetic vortices" (m particles). It is shown that these particles have unusual mutual statistical properties, as we can get a phase flip -1 if we move one particle around the other, which are thus called Abelian 1/2 anyons [5] [see Fig. 1(b)].

Recently, Han, Raussendorf, and Duan [14] exploited the fact that the statistical properties of anyons are manifested by the underlying ground and excited states [17]. So, instead of direct engineering the interactions of the Hamiltonian H (2) and ground-state cooling which are extremely demanding experimentally, an easier way is to dynamically create the ground state and the excitations of this model Hamiltonian, encoding the underlying anyonic model in a multiparticle entangled state which can be used to simulate the dynamics of the anyonic system. The quasiparticles are then defined by the negative outcome of a stabilizer element A_v or B_f. Specifically, as illustrated in Fig. 1(b), with the ground state $|\psi_g\rangle$ prepared, one can first create a pair of e particles by applying a single-qubit Z rotation. The system wave function will be in the excited state $|\psi_e\rangle$. To make fractional phase experimentally detectable in a later stage, we apply a $\sqrt{Z}$ rotation and get a superposition state $(1/\sqrt{2})(|\psi_g\rangle + |\psi_e\rangle)$. Then, we create another pair of m particles and move one of them around an e particle along a closed loop, and finally annihilate the m particles. After doing so, it is predicted that a fractional phase π will be added to $|\psi_e\rangle$; thus, the superposition state will become $(1/\sqrt{2})(|\psi_g\rangle - |\psi_e\rangle)$.

As the anyons are perfectly localized quasiparticles in this model Hamiltonian, a small spin lattice containing six qubits shown in Fig. 2(a) allows for a proof-of-principle experimental demonstration [5,14]. The Hamiltonian of this model is $H_6 = -A_1 - A_2 - B_1 - B_2 - B_3 - B_4$, where $A_1 = X_1X_2X_3$, $A_2 = X_3X_4X_5X_6$, $B_1 = Z_1Z_3Z_4$, $B_2 = Z_2Z_3Z_5$, $B_3 = Z_4Z_6$, $B_4 = Z_5Z_6$ (the subscripts of the Pauli matrices label the qubits). The ground state $|\psi\rangle_6$ of this Hamiltonian H_6 is locally equivalent to a six-qubit graph state [18,19], which can be represented by the graph as depicted in Fig. 2(b). This equivalence follows from the fact that the operators $A_1, \cdots, B_4$ in the Kitaev model can be uniquely derived from the stabilizer operators g_i [see Fig. 2(b)] of the graph state.

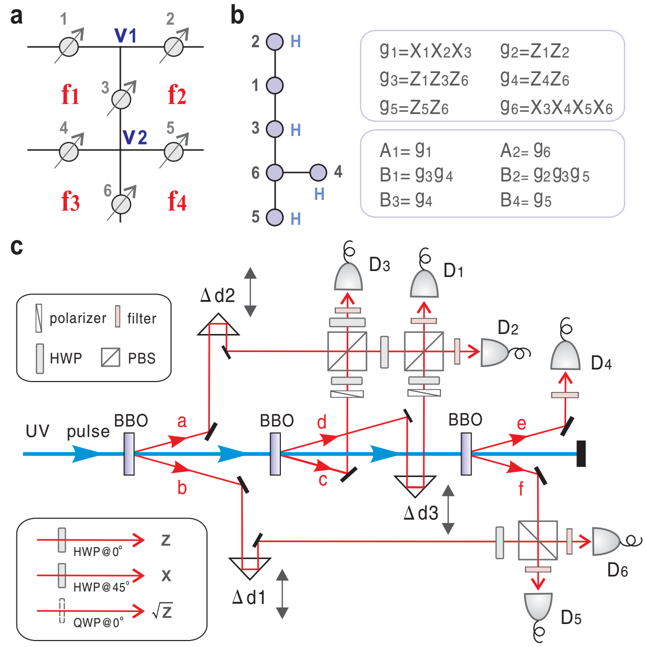

FIG. 2 (color online). (a) The small Kitaev spin lattice system with six qubits used for demonstration of anyonic braiding operations. (b) The six-qubit graph state which, after Hadamard (H) transformations on qubit 2, 3, 4, and 5, is equivalent to the ground state of the system in Fig. 2(a). The graph state is associated with a graph, where each vertex denotes a qubit and each edge represents a controlled phase gate having been applied between the two connected qubits [18,19]. The graph state is a common eigenstate of the stabilizer operators g_i, that is, $g_i|\psi\rangle_6 = |\psi\rangle_6$, which describe the correlation in the state, and the graph state is the unique state fulfilling this. Here, we use the same label as Figs. 2a-b in Ref. [14]. (c) Experimental setup. A ultraviolet laser successively passes through three β-barium borate (BBO) crystals to generate three pairs of entangled photons [20]. The photons a, b, c, d, and f are combined on the three polarizing beam splitters (PBSs) step by step [27]. To achieve good spatial and temporal overlap, all photons are spectrally filtered ($\Delta\lambda_{\mathrm{FWHW}} = 3.2$ nm) and detected by fiber-coupled single-photon detectors ($D_1, \cdots, D_6$). The detector labels correspond to the qubit labels in the text and in Figs. 2(a) and 2(b). The coincidence events are registered by a programmable multichannel coincidence unit. For single-qubit rotations and polarization analysis, quarter wave plates (QWPs), half wave plates (HWPs), together with polarizers or PBSs are used.

Now we proceed with the experiment in three steps: (1) create the ground state $|\psi\rangle_6$, (2) verify the anyonic excitations, (3) implement the braiding operations and detect the phase shift. We use single photons as a real physical system to simulate the creation and control of the anyons. The quantum states are encoded in the polarization of the photons which are robust to decoherence. The experimental setup is illustrated in Fig. 3(c). We start from three pairs of entangled photons produced by spontaneous parametric down-conversion (SPDC) [20]. The photons in spatial modes a-b and e-f are prepared in the states

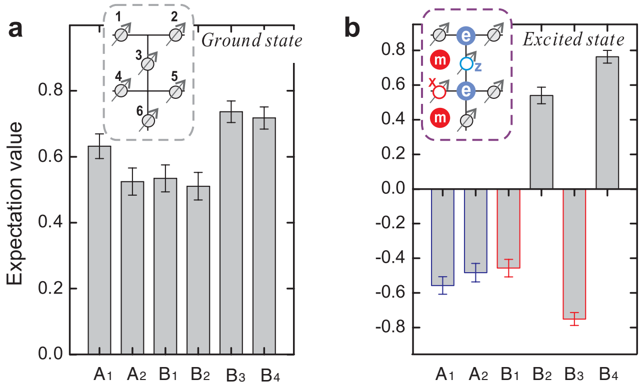

FIG. 3 (color online). The measured expectation values of the operators $A_1, \cdots, B_4$ of the ground state $|\psi\rangle_6$ (a) and the excited state $|\psi_{em}\rangle_6$ (b). The excited state $|\psi_{em}\rangle_6$ has a pair of e particles on v_1, v_2 and a pair of m particles on f_1, f_3; thus, the values for A_1, A_2, B_1, B_3 become negative. Each expectation value is derived from a complete set of 64 sixfold coincidence events in 15h in measurement basis $Z^{\otimes 6}$ or $X^{\otimes 6}$. The error bars represent 1 standard deviation, deduced from propagated Poissonian statistics of the raw detection events.

$|\phi^+\rangle_{ij} = (1/\sqrt{2})(|H\rangle_i|H\rangle_j + |V\rangle_i|V\rangle_j)$, while those in mode c-d are disentangled using polarizers and then prepared in the states $|+\rangle_i = (1/\sqrt{2})(|H\rangle_i + |V\rangle_i)$, where $H(V)$ denotes horizontal (vertical) polarization, and i and j label the spatial modes. We then pass the photons through a linear optics network [see Fig. 3(c)]. A coincidence detection of the six outputs corresponds exactly to the ground state

$$\begin{aligned}|\psi\rangle_6 = \frac{1}{2}(&|H\rangle_1|H\rangle_2|H\rangle_3|H\rangle_4|H\rangle_5|H\rangle_6 \\ &+ |V\rangle_1|V\rangle_2|V\rangle_3|H\rangle_4|H\rangle_5|H\rangle_6 \\ &+ |H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4|V\rangle_5|V\rangle_6 \\ &+ |V\rangle_1|V\rangle_2|H\rangle_3|V\rangle_4|V\rangle_5|V\rangle_6). \qquad (3)\end{aligned}$$

To verify that the ground state $|\psi\rangle_6$ has been obtained, first we experimentally measure the expectation values of its stabilizer operators $A_1, \cdots, B_4$. These stabilizer operators describe the intrinsic correlations in the state $|\psi\rangle_6$ and uniquely define it; thus, their expectation values could serve as a good experimental signature. For an ideal state $|\psi\rangle_6$, all expectation values should give $+1$. In our experiment, however, the ground state was created imperfectly. Figure 3(a) shows the measurement results, with all expectation values being positive in a rang between 0.51 ± 0.04 and 0.74 ± 0.03, in qualitative agreement with the theoretical prediction. For a more complete and quantitative analysis, we aim to estimate the fidelity of the produced state, that is, its overlap with the desired one. This quantity is given by $F_{\psi_6} = {}_6\langle\psi|\rho_{\rm exp}|\psi\rangle_6$, which is equal to one for an ideal state, and $1/64$ for a completely mixed six-qubit state. To do so, we consider a special observable, which allows for lower bounds on the fidelity, while being easily measurable with few correlation measurements [21,22]. By making these measurements, we estimate the fidelity of the created ground state to be $F_{\psi_6} \geq 0.532 \pm 0.041$ [23]. The imperfection of this state is mainly caused by the high-order photon emissions during the SPDC and the partial distinguishability of independent photons.

We now move to the step (2). With the ground state $|\psi\rangle_6$ created, we apply a Z (X) rotation on qubit 3 (4), creating an excited state $|\psi_{em}\rangle_6$ on which a pair of e particles live on the vertices v_1 and v_2, and another pair of m particles on faces f_1 and f_3 [see Fig. 2(a)]. The Z and X rotations are experimentally realized using HWPs oriented at 0° and 45°, respectively. As discussed before, the anyonic excitations are signaled by violations of stabilizer conditions, that is, $A_{v_i}|\psi_{em}\rangle_6 = -|\psi_{em}\rangle_6$, $B_{f_i}|\psi_{em}\rangle_6 = -|\psi_{em}\rangle_6$ [5]. Thus, in our case, theoretically, the expectation values of A_1 and A_2 should become -1 because of the e particles, and the same for B_1 and B_3 due to the m particles. To verify this, we measure the expectation values of the operators $A_1, \cdots, B_4$. The results are shown in Fig. 3(b), where the values of A_1, A_2, B_1, and B_3 flip compared to those shown in Fig. 3(a) which supports the presence of anyonic excitations [5,14].

Now, we proceed to step (3). On the ground state $|\psi\rangle_6$, first we apply a $\sqrt{Z}$ operation using a QWP oriented at 0° on the qubit 3 of the ground state $|\psi\rangle_6$, yielding a superposition state $|\psi_s\rangle_6 = (1/\sqrt{2})(|\psi\rangle_6 + |\psi_e\rangle_6)$, where $|\psi_e\rangle_6$ is the excited state with a pair of e particles on v_1 and v_2. With an X rotation on the qubit 4, we further create a pair of m particles on f_1 and f_3. Then, we perform four X operations on the qubits 6-5-3-4 to implement the braiding operation, that is, the m particle on f_3 is moved around the e particle on v_2 along an anticlockwise closed loop. We note that the crossing at the qubit 3, which is unavoidable in two dimensions, is relevant for the unusual statistics. Finally, the pair of m particles is annihilated with an X operation on qubit 4 (fusion).

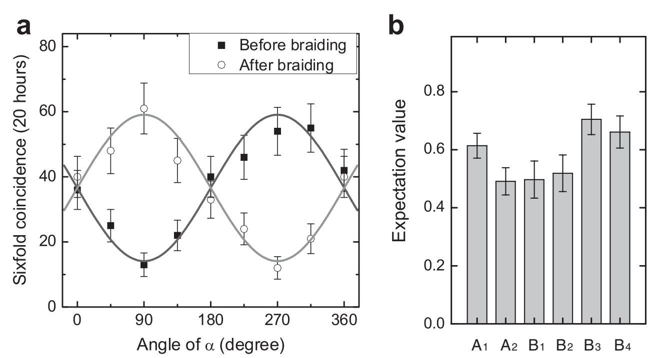

FIG. 4. (a). Measured fringes for the state $|\psi_s\rangle_6$ and $|\psi_f\rangle_6$. Two sinusoid curves are fitted to the data with visibilities of 0.61. The measurements in basis $(|+\rangle + e^{i\alpha}|-\rangle)$ are done using a combination of HWPs, QWP, and PBS. (b). The expectation values of the operators $A_1, \cdots, B_4$ of the state $|\psi_f\rangle_6$ after the $\sqrt{Z_3}$ transformation.

After these, if there is a fractional phase ϕ acquired, the state $|\psi_s\rangle_6$ will become $|\psi_f\rangle_6 = (1/\sqrt{2})(|\psi\rangle_6 + e^{i\phi}|\psi_e\rangle_6)$. To determine this ϕ, we look at the correlation measurement outcomes of the six photons where the photons 1 and 2 are fixed at $|+\rangle$ polarization, 4, 5, and 6 at $|H\rangle$, and the photon 3 is measured in the basis $(|+\rangle + e^{i\alpha}|-\rangle)$ with α varying in $\pi/4$ steps. In this setting, the sixfold coincidence counts should follow the relation $C(\phi,\alpha) \propto 1 + \sin(\phi - \alpha)$ for the state $|\psi_f\rangle_6$; thus, an unknown phase ϕ, if it occurs, can be revealed. Figure 4(a) shows the measurement results for both the state $|\psi_s\rangle_6$ and $|\psi_f\rangle_6$, before and after the process of m particle creation, braiding, and fusion. These two curves clearly exhibit a phase difference of π, confirming the prediction of the fractional statistics.

For a more complete proof, we implement a $\sqrt{Z}$ transformation on the qubit 3 of the remaining state $|\psi_f\rangle_6$. The state $|\psi_f\rangle_6$ will be converted to $|\psi\rangle_6$ if there is a fractional phase π; otherwise, it will go to $|\psi_e\rangle_6$. To test this, again we measure the expectation values of the operators $A_1, \cdots, B_4$ of the state after the $\sqrt{Z_3}$ transformation. The experimental results are shown in Fig. 4(b), which are in agreement with that the final state is $|\psi\rangle_6$ and thus prove the fractional phase change of $\phi = \pi$. Here, we note that the facts that the present setup use free-flying noninteracting photons and that the time scale of the braiding operations is extremely small ($\sim$ picoseconds for photons passing through the HWPs and QWPs) implies that this acquired phase cannot be a dynamical phase. Moreover, the creating, braiding, and annihilating of the m particles which corresponds to the operation $X_4X_6X_5X_3$ do not give rise to a phase either, as the ground state $|\psi\rangle_6$ is an eigenstate of this observable $X_4X_6X_5X_3$. Similar arguments also apply to the e particles' case. Consequently, this excludes possible geometrical phases [24] and proves the observed phase is purely statistical.

In summary, we have demonstrated the creation and manipulation of anyons in the Kitaev spin lattice model, and observed the fractional statistics of the Abelian 1/2 anyons. This has been done without generating the four-body interactions in the model Hamiltonian but alternatively, in an easier way—by encoding the underlying anyonic system on a six-photon graph state, or equivalently, realizing the six-qubit circuit shown in Fig. 2c of Ref. [14]. It should be noted that the absence of the Hamiltonian does not prevent us from studying the topological and statistical properties of the anyons here; however, topological quantum computing in a fault-tolerant manner would eventually require such a Hamiltonian. From a quantum-information perspective, our experiment may be seen as using quantum computers which have already well understood physics as a tool to simulate other difficult quantum systems. Such quantum simulators can in principle provide exponential speedup in the simulation of quantum physics [25], and may offer a more controllable and clean access to study strongly correlated behaviors than natural complex solid-state systems. Possible future work along this line may include investigation of the robustness of anyonic braiding [14] and other possible phases [24], and demonstration of some basic elements of cluster-state topological quantum computing [26].

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